THE MECHANICAL PROPERTIES OF FLUIDS

A Collective Work by

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PUBLISHERS' NOTE

In recent years a great many researches have been made into the mechanical properties of fluids by physicists and engineers. These researches are of the utmost practical importance to engineers and others, but it is not unusual to find that the people who are called upon to apply the results in industry have considerable difficulty in finding connected accounts of the work. It is hoped that this collection of essays, many of which are written by men who are the actual pioneers, will prove of use in making the recent discoveries in the mechanical properties of fluids more generally known. The mathematical notation has been made uniform, and the different chapters have been collated as far as possible.

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INTRODUCTION

By Engineer Vice-Admiral
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(Late Engineer-in-Chief of the Fleet)

To those engaged in the practice of engineering, and able and willing to utilize the information that mathematical and physical science can offer them, it is of great assistance to have such information readily available in direct and relevant connection with the problems with which they are confronted.

The collective work of this book, issuing as it does from authors highly qualified and esteemed in their respective fields, whose views and statements will be accepted as authoritative, supplies concisely and consistently valuable information respecting the mechanical properties of fluids, and elucidates the evolution of many successful practical applications from first considerations.

Much has been done in this direction in regard to solids, and this has been assimilated and usefully applied by many; but much less has been produced on the subject of fluids, especially in compact form, and this collective work will doubtless on this account be very welcome.

The necessities of the war brought us face to face with many new problems, a large number of which required not only prompt application of the knowledge available,

but intensive research and rapid development in order to comply with the constantly increasing standards of quality that were demanded. Most of the results are well known: the principles by which they were reached, especially in regard to fluids, are perhaps only vaguely understood, except by a few. The results are certainly appreciated, but further application is probably hindered in many directions for want of this knowledge.

The several contributions to this work enunciate clearly the principles involved, and indicate that a wide field is open for the application of these principles to those who are engaged in industrial avocations and pursuits, as well as to others whose duties continue to be confined exclusively to preparation for war.

Chapters dealing mainly with theoretical considerations form a prelude, clarifying ideas of the physical properties of fluids and providing a sketch of the mathematical theory of fluid motion, with indicators to practical utility.

The sections devoted to practical applications are developed from the underlying theoretical and mathematical considerations. The retention of this method of treatment of the several subjects throughout the work is a valuable feature. These sections will interest a variety of readers, some parts being of particular value to specialists such as the gunnery expert, the naval architect, and the aeronautical engineer, but by far the greater portion of the book will be of general interest to the large body of engineers who have to deal with or use fluids for many purposes in their everyday work.

The chapter on viscosity and lubrication should point the way to the better appreciation and further application of the correct principles of lubrication. The best known present application, that of enabling propeller thrust loads of high intensity to be taken on a single collar, has been highly successful, and it is gratifying to observe that other develop-

ments are already in contemplation, and that some are well advanced.

The description of the determination of stress by means of soap films is fascinating and deeply interesting; and it is cheering to know that certain forms of stress in members of irregular form under load, not amenable to calculation, and hitherto not determinable and therefore provided for by a factor of safety, can now be closely approximated to by experimental means, and it may be hoped that this or some other experimental process can be extended practically in the near future to determine other forms of stress. Success in this direction would be directly attended with economy of material and would facilitate design.

The chapter on submarine signalling indicates the march of progress in a new branch of engineering, and the author makes the important and significant remark that the science of acoustics shows signs of developing into the engineering stage, a statement worthy of the careful consideration of all thoughtful engineers.

The section dealing with the wave transmission of energy comes opportunely in view of the large number of practical applications of this form of power transmission that are being developed, and of those that have matured. The same section gives information respecting the principles governing the various forms of flow-meters, and should prove useful to engineers associated with high-power installations who are, by reason of the magnitude of the individual installations, being forced to use flow measurements in lieu of the definite bulk measurements hitherto favoured by many, and should give a greater confidence in the use and accuracy of flow-meters designed on a sound basis.

The preceding cases are merely mentioned as examples; every chapter contains a great deal of matter of practical topical engineering interest connected with the mechanical properties of fluids. I have selected these examples as some

of those familiar to me in which I have personally felt the want of some preparatory and explanatory information, such as that given in this book; and it is my recollection that such information was more difficult or inconvenient to obtain in regard to fluids than for solids. My own experiences must, I feel, be those of many others.

The whole series of articles has been to me most interesting, and they show clearly that engineering in the present day requires a great deal of help from pure and experimental science, and is adapting itself to the utilization of branches of science with which it has hitherto not been closely associated. Engineering practice to be worthy of the name must keep itself abreast of and well in touch with those sciences and the developments and discoveries connected with them. This is an onerous task and can only be effected collectively; it is too big for one individual; but works such as this will tend to ease the burden, and convert the task into a pleasing duty.

LIST OF SYMBOLS USED

p or P, pressure.
v or V, volume; velocity.
N, modulus of rigidity.
E, Young's modulus.
t, temperature °C. (or °F.); time.
ρ, density.
s, specific gravity.
κ, bulk modulus; eddy conductivity.
T, absolute temperature in °C.
τ, absolute temperature in °F.
C_v, specific heat at constant volume.
C_p, specific heat at constant pressure.
γ, surface tension.
β, compressibility.
M, molecular weight.
m, mass.
μ, viscosity.
ν, kinematic viscosity $\equiv \frac{\mu}{\rho}$.
u, v, w, component velocities; displacements.
N, Avogadro's constant.
ω, specific volume of water; cross section; angular velocity.
S, shearing stress.
$\mathfrak{T}$, twist.
T_q, torque.
λ, wave-length; film energy.
R, acoustic resistance.
$\sim$, frequency (cycles per second).

THE MECHANICAL PROPERTIES OF FLUIDS

CHAPTER I

Liquids and Gases

Definitions

We propose to discuss in this chapter some of the more important general properties of fluids. Common knowledge enables us to associate with the terms *fluid*, *liquid*, *vapour*, *gas*, certain properties which we regard as fundamental, and which serve to differentiate these forms of existence from the form which we know as *solid*. A fluid is, etymologically and physically, *that which flows*, and the liquid or the gaseous state is a special case of the fluid form of existence. A liquid, in general, is only slightly compressible and possesses one free bounding surface when contained in an open vessel. A gas, on the other hand, is easily compressible under ordinary circumstances, and always fills the vessel which contains it.* The most elementary observation forces upon our notice distinctions such as these just mentioned, but it still remains to be seen whether these can be made the basis of a satisfactory classification.

Indeed, it is doubtful whether we can make a classification which will conveniently pigeon-hole the different states of matter, for, as we shall see in the sequel, these different states shade over, under special circumstances, one into the other, without the slightest breach of continuity.

Ordinarily the change from solid to liquid—as when ice becomes

* But compare the quotation on p. 2.

water—or from liquid to vapour—as when water boils—is quite sharp, and the properties of any one substance in the three states are clearly marked off. But substances such as pitch or sealing-wax—behaving under some circumstances as solids, under others as liquids—are distinctly troublesome to the enthusiast for classification. Thus a bell or tuning fork, cast from pitch, will emit a note perfectly clear and distinct as that given by a bell of metal. Nevertheless a block of pitch, left to itself, will in time flow like any ordinary liquid. Steel balls placed on the top of pitch contained in a vessel slowly sink to the bottom, and corks placed at the bottom of the vessel will in time appear at the upper surface of the pitch. Such anomalies serve to emphasize the difficulties attendant on any attempt at a rigorous classification. Indeed it is sometimes held that the difference between the solid and liquid states is one of degree, and that all solids in some measure show the properties of liquids. However this may be, it is enough to note now that the differences between the solid, liquid, and gaseous states are sufficiently pronounced to make it convenient to attempt a classification which shall emphasize these differences. We shall therefore discuss certain properties of matter which serve to define *ideal* solids, liquids, and gases. We shall find that no substances in nature conform to our ideal, which will therefore be but a first approximation to the truth, and later we shall find that small corrections, applied to the equations of state which are the expression of our fundamental definitions, will serve to make the equations represent with considerable accuracy the behaviour of actual substances. This process, involved though it may appear, is both historically correct and physically convenient.

Thus the reader may remember that in 1662 the Honourable Robert Boyle took a long glass tube "which by a dexterous hand and the help of a lamp was in such a manner crooked at the bottom that the part turned up was almost parallel to the rest of the tube, and the orifice of this shorter leg . . . being hermetically sealed, the length of it was divided into inches. . . . Then putting in as much quicksilver as served to fill the arch, . . . we took care, by frequently inclining the tube, so that the air might freely pass from one leg into the other, . . . (we took, I say, care) that the air at last included in the shorter cylinder should be of the same laxity with the rest of the air about it. This done, we began to pour quicksilver into the longer leg, . . . till the air in the shorter leg was by condensation reduced to take up but half the space it possessed (I say *possessed* not *filled*) before; we cast our eyes upon the longer leg of the glass, . .

and we observed, not without delight and satisfaction, that the quicksilver in that longer part of the tube was 29 in. higher than the other."*

The pressure and volume of a gas at constant temperature are therefore in reciprocal proportion; that is, at constant temperature the equation of state of a gas is given by

$$pv = k.$$

Succeeding experiments emphasized the truth of this result, and it was not until instrumental methods had advanced considerably that small deviations from this law were shown to exist under ordinary conditions. It was then proved that an equation of the type

$$\left(p + \frac{a}{v^2}\right)\left(v - b\right) = k$$

more closely represented the behaviour of even the more permanent gases; later work has shown this equation does not represent with sufficient accuracy the results of experiment, and various other equations of state have, from time to time, been proposed. To these equations we shall later have occasion to refer.

Again, the *physical* convenience of such a method of approach may be illustrated by results deduced from the principles of rigid dynamics. No body in nature is perfectly rigid—that is, is such that a line joining *any* two particles of the body remains invariable in length during the motion of the body—but considerable simplification of the equations of motion results if we make this assumption, and the results obtained are in many cases of as high an order of accuracy as is required. We can, if necessary, obtain a closer approximation to the truth by considering the actual deformation suffered by the body—the problem then becoming one in the theory of elasticity. Suppose, for example, that it is our object to deduce the acceleration due to gravity from observation of the period of a compound pendulum. It would be possible to attack the problem, taking into account *ab initio* such effects as are due to, say, deformation of the pendulum in its swing, yielding of the supports and the like, afterwards neglecting such effects as experience has shown to be very small. But such a method would render the problem almost unbearably complex, besides tending to distract attention from the essentials, and it is both more convenient and more philosophic to focus one's mind on the more important issues, to solve the problem first for an ideal

* Boyle's works (Birch's edition, Vol. I, p. 156, 1743).

rigid body, and afterwards to introduce as small corrections effects due to elasticity, viscosity, and so forth.

This, then, is the course which we shall follow in discussing the properties of fluids, and we shall seek, using elastic properties as a guide, definitions which will emphasize those differences which undoubtedly exist between the solid, liquid, and gaseous states of existence.

If we wish to describe completely the elastic behaviour of a crystalline substance, we find that in the most general case twenty-one coefficients are required. For isotropic substances, fortunately, the problem is much simpler, and the coefficients reduce to two, the bulk modulus (κ) and the rigidity modulus (N). These coefficients are

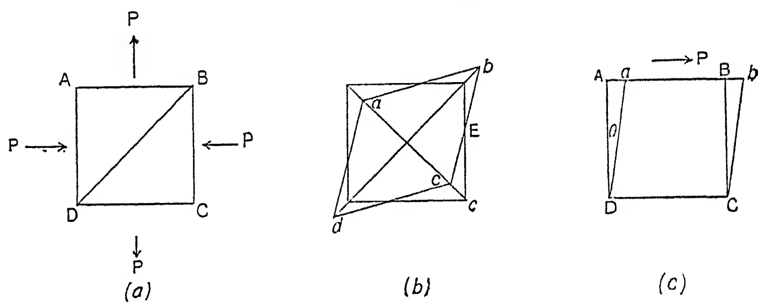


Fig. 1

easily specified. Thus, if a cube of unit edge be subjected to a uniform hydrostatic pressure P, so that its volume decreases by an amount δv, the sides of the cube decreasing by an amount e; then, δv being the change in volume per initial unit volume, the ratio of stress to strain, which is the measure of the bulk modulus, is given by $\frac{P}{\delta v} = \kappa$, and, to the first order of small quantities, $\delta v = 3e$.

Suppose now that our unit cube is strained in such a way that in one direction the sides are elongated by an amount e, in a perpendicular direction are contracted by an amount e (fig. 1 a), the sides perpendicular to the plane of the drawing being unaltered in length. Such a strain is called a shearing strain, and may be supposed to be produced by stresses (P) acting as shown. Considering the rectangular prism BCD, which is in equilibrium under the stresses acting normally over the faces BC and CD, and the forces due to the action on BCD of the portion ABD of the cube, we see that the resultant of the two forces P is a force $P\sqrt{2}$ acting along BD. The

force due to the action of ABD on BCD must be equal and opposite to this. But the area of the face BD is $\sqrt{2}$ units, and there is therefore a tangential *stress* of P units, in the sense DB, acting over the diagonal area of the cube, due to the action on the prism BCD of the matter in the prism ABD, and called into being by the elastic displacements. Thus *the shearing stress, which produces the shearing strain, may be measured by the stress on the areas of purely normal or of purely tangential stress.*

If we suppose the directions of the principal axes of shear to be along the diagonals AC and BD, so that these diagonals are contracted and elongated respectively by an amount e (per unit length), then it can easily be shown that, assuming the strains to be small, the side of the square, the area of the square, and the perpendicular distance between its sides are, to the first order of small quantities, unaltered by the strain. Hence (fig. 1 *b*) the square ABCD strains into the rhombus *abcd*, and by rotating the rhombus through the angle *c*EC—which rotation does not involve the introduction of any *elastic* forces—we arrive at the state shown in fig. 1 *c*. Hence, rotation neglected, a shearing strain may be regarded as being due to the sliding of parallel planes of the solid through horizontal distances which are proportional to their vertical distances from a fixed plane DC, the sliding being brought about by a tangential stress P applied to the plane AB. The angle θ is taken as a measure of the strain, and the rigidity modulus (N) is given by the equation

$$N = \frac{P}{\theta}.$$

It is to be remembered that an elastic modulus such as Young's modulus (E) is not independent of κ and N, but is connected with them, as can readily be proved, by the relation

$$E = \frac{9\kappa N}{3\kappa + N}.*$$

We are now in a position to define formally the terms "solid" and "fluid".

A solid possesses both rigidity and bulk moduli. If subjected to shearing stress or to hydrostatic pressure it takes up a new position of equilibrium such that the forces called into existence by the elastic displacements form, with the external applied forces, a system in equilibrium.

* Tait, *Properties of Matter*, p. 155, or Morley, *Strength of Materials* (1921), p. 11.

A fluid possesses bulk elasticity, but no rigidity. It follows, therefore, that a fluid *cannot permanently resist a tangential stress*, and that, however small the stress may be, the fluid will, in time, sensibly yield to it. In a solid, the stress on an element-plane may have any direction with reference to that plane. It may be purely normal, as on the plane BC (fig. 1 *a*), or purely tangential, as on the plane BD. In a fluid *at rest* the stress on an element-plane must be normal to that plane. And it follows at once from this normality, as is proved in all elementary treatises on hydrostatics, that the pressure (p) at a point in a fluid at rest under the action of any forces is independent of the orientation of the element-plane at that point. Thus if x, y, z are the co-ordinates of the point in question,

$$p = \phi(x, y, z).$$

In a *perfect fluid*, no tangential stresses exist, whether the fluid be at rest, or whether its different parts be in motion relative to each other. In all fluids known in nature, tangential stresses tending to damp out this relative motion do exist, persisting as long as the relative motion persists. The fluid may be looked on as yielding to these stresses, different fluids yielding at very different time-rates; the *rate* of yield depends on the property known as viscosity.

A *perfect liquid* may be defined as an incompressible perfect fluid. No fluid in nature is completely incompressible, and the quantitative study of the bulk moduli of liquids and their relation to other constants of the liquid substance is a matter of great theoretical and practical importance.

It must be remembered that the magnitude of the bulk modulus depends on the conditions under which the compression is carried out. Two moduli are of primary importance—that in which the temperature of the substance remains constant, and that in which the compression is adiabatic, so that heat neither enters nor leaves the substance under compression. Remembering this, we may define a *perfect gas* as a substance whose bulk modulus of isothermal elasticity is numerically equal to its pressure. From this we have at once by definition

$$\frac{dp}{-\frac{dv}{v}} = p,$$

or $$pdv + vdp = 0.$$

Whence, integrating,

$$pv = k,$$

and our perfect gas follows Boyle's Law.

Density

Having obtained working definitions of the substances with which we have to deal, we proceed to discuss in order certain of their more fundamental properties and constants. One of the most important of these constants is the *density* of the fluid, defined as the *mass* of unit volume of the fluid. The density of a liquid is accepted, in a chemical laboratory, as one of the tests for its identification, and the importance in industry of the "gravity" test needs no emphasis. We shall therefore detail one or two methods for the measurement of the density of a liquid—methods for the measurement of gaseous or vapour densities are perhaps more appropriately discussed in a treatise on heat.

To determine the density of a substance we have to measure either (*a*) the mass of a known volume or (*b*) the volume of a known mass. Fluids must be weighed in some sort of containing vessel, and if we know the volume of the containing vessel, the measurement of the mass of fluid which fills it at a given temperature at once gives us the density of the fluid. The most convenient way of calibrating a containing vessel is by finding the weight of some liquid of known density which fills the vessel at a known temperature. This assumes, of course, that the density of the standard liquid—usually water or mercury—has been determined by some independent method, and much laborious research has been done on the measurement of the variation of density with temperature of these two fluids.

Thus, Hälstrom * measured carefully the linear expansion of a glass rod, the relation between length and temperature being expressed by the formula

$$L = L_0(1 + at + bt^2).$$

A piece of this rod of volume V was taken and weighed in water at different temperatures, the *loss* in weight in water at t° being given by

$$W = W_0(1 + lt + mt^2 + nt^3).$$

* *Ann. Chim. Phys.*, 28, p. 56.

The quantities a, b, l, m, n are determined by experiment, and it is clear that the volume at t° of the portion used is

$$V = V_0 (1 + at + bt^2)^3.$$

Now since the loss of weight in water is, by Archimedes' principle, equal to the weight of water displaced, and since the volume of this displaced water is equal to the volume of the glass sinker, we have for the density of water at t°

$$\rho = \frac{W}{V} = \frac{W_0}{V_0} \cdot \frac{(1 + lt + mt^2 + nt^3)}{(1 + at + bt^2)^3},$$

or

$$\rho = \rho_0(1 + \alpha t + \beta t^2 + \gamma t^3),$$

where α, β, γ are known in terms of l, m, n, a, b. The figures obtained in an actual experiment are quoted below:

$$\left.\begin{array}{rl} a = & 0{\cdot}000001690 \\ b = & 0{\cdot}000000105 \\ l = & 0{\cdot}0000058815 \\ m = & -\,0{\cdot}0000062168 \\ n = & 0{\cdot}00000001443 \end{array}\right\}\text{whence}\left\{\begin{array}{rl} \alpha = & 0{\cdot}000052939 \\ \beta = & -\,0{\cdot}0000065322 \\ \gamma = & 0{\cdot}00000001445 \end{array}\right.$$

It may be noted in passing that the temperature of maximum density of water may be determined from these results with considerable accuracy. For when ρ is stationary we have $\frac{d\rho}{dt} = 0$, and hence

$$3\gamma t^2 + 2\beta t + \alpha = 0.$$

This equation is a quadratic in t; one of the roots is outside the range of the experimental figures, the other is 4·108° C.

From experiments of which this may be quoted as a type, Table I (p. 9) has been drawn up.

It will be seen that, if water be used as the calibrating liquid, the determination of the density of a liquid becomes identical with the operation of determining its specific gravity—that is, we find by experiment the ratio of the weight of a certain volume of liquid to that of an equal volume of water *at the same temperature*. The magnitude of this ratio is conveniently denoted by the symbol s_t^t, and may be reduced to density—mass per cubic centimetre—by means of Table I on p. 9. It is more usual to compare the

TABLE I

DENSITY OF WATER IN GM./C.C. AT VARIOUS CENTIGRADE TEMPERATURES

Temperature.	Density.	Temperature.	Density.
Degrees.		Degrees.	
0	0·99987	42	0·99147
2	0·99997	44	0·99066
4	1·00000	46	0·98982
6	0·99997	48	0·98896
8	0·99988	50	0·98807
10	0·99973	52	0·98715
12	0·99953	54	0·98621
14	0·99927	56	0·98525
16	0·99897	58	0·98425
18	0·99862	60	0·98324
20	0·99823	62	0·98220
22	0·99780	64	0·98113
24	0·99732	66	0·98005
26	0·99681	68	0·97894
28	0·99626	70	0·97781
30	0·99567	75	0·97489
32	0·99505	80	0·97183
34	0·99440	85	0·96865
36	0·99371	90	0·96534
38	0·99299	95	0·96192
40	0·99224	100	0·95838

weight of the liquid with that of an equal volume of water at 4° C., and this value, s_4^t, may also be deduced from the experimental figures by means of the table. It should be noted that the specific gravities s_t^t and s_4^t are often doubtful in meaning, for they refer sometimes to the ratio of true weights, sometimes to the ratio of apparent weights, no correction being made for displaced air. In experimental work of high accuracy, it is well both to make this correction, *and to indicate that it has been made.*

For ordinary work, the common specific-gravity bottle may be used, but for precision measurements some form of pyknometer is necessary. The pyknometer is usually a U-tube of small cubic content, its ends terminating in capillary tubes. Three forms are outlined in fig. 2. (i) is the original Sprengel type. (ii), a modification introduced by Perkin, possesses several advantages. The instrument, filled by suction, is placed in an inclined position in a thermostat, and excess

liquid is withdrawn from *a* by means of filter paper, until the level in the other limb falls to *b*. The tube is now removed and restored to the vertical position, when the liquid recedes from *a*. If now expansion takes place before weighing, the bulb above *b* acts as a safety space, and all danger of loss by overflow is obviated. The form shown in (iii) was introduced by Stanford, and reduces to a minimum those parts of the vessel not contained in the thermostat, whilst its shape does away with the necessity for suspending wires, as the bottle can be weighed standing upon the balance pan.

In technological practice much specific-gravity work is carried out by means of variable-immersion hydrometers. Hydrometer practice and methods can hardly be said to be in a satisfactory state.

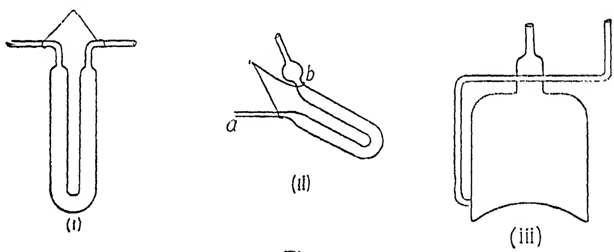


Fig. 2

Not only has one to plough through a jungle of arbitrary scales, but the reduction of these scale readings to specific gravities, defined *accurately* as we have defined them already, is no easy matter. All hydrometers should carry, marked permanently on their surfaces, some indication of the principle of their graduation, so that their readings may be reduced to s_t^t, s_4^t, or some other definitely known standard. For rough work, of course, the arbitrary graduations suffice, and a workman soon learns to associate a reading of, say, x degrees Twaddell with some definite property of the liquid with which he is working. But with more delicate hydrometers an absence of *exact* reference to some definite standard is distinctly unsatisfactory.

Thus the common hydrometer is graduated so that a reading of 1035 corresponds to a specific gravity of 1·035—the standard of reference being very often doubtful—and the Twaddell hydrometer is so constructed that the specific gravity s is given by

$$s = 1{\cdot}000 + 0{\cdot}005\,\tau,$$

where τ is the reading in Twaddell degrees. Clearly on the common

hydrometer the "water-point" is 1000, on the Twaddell hydrometer zero, and, unless the hydrometer carries some reference to the temperature at which its water-point is determined, it becomes impossible satisfactorily to compare the performances of two different hydrometers.

Confusion is worse confounded when we introduce Baumé readings. In the original Baumé hydrometer water gave the zero point, and a 15 per cent solution of sodium chloride gave the 15° mark. This for liquids heavier than water. For liquids lighter than water a 15 per cent solution of salt marked the zero point, the water-point being at 10°. Now it is usual to mark the point to which the hydrometer sinks in sulphuric acid of density 1·842 as 66° B. We can easily work out a formula of reduction giving the specific gravity in terms of these fixed points. Thus, let

V be the total volume of hydrometer up to 0°,
ρ_1, the density of water,
v, the volume of hydrometer between 0° and n°,
ρ, the density of liquid in which hydrometer floats at mark n, and
a, the cross-sectional area of neck.

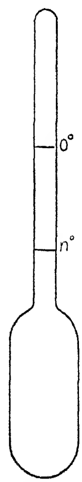


Fig. 3

By Archimedes' principle the mass of the hydrometer is given by the two expressions

$$V\rho_1 \text{ and } (V - v)\rho.$$

Hence

$$V\rho_1 = (V - v)\rho = (V - an)\rho;$$

and if s is the specific gravity of the liquid,

$$s = \frac{\rho}{\rho_1}.$$

Hence we have

$$\frac{ns}{s - 1} = \frac{V}{a} = k.$$

If we put $s = 1·842$ when $n = 66$, we find that $k = 144·3$, and therefore for any other liquid giving a reading x,

$$s = \frac{144·3}{144·3 - x}.$$

It is obvious that we do not know where we are unless the densities

used in calibration are sharply defined, and the clouds are not appreciably lightened by the practices of Dutch and American hydrometer makers, who take the constant k as 144 and 145 respectively—presumably for the convenience of dealing in integral numbers.

Mohr's balance is exceedingly convenient for use in those technological laboratories in which a large number of determinations of density are made. As fig. 4 shows, it is a balance of special form, one arm being divided into ten equal parts and carrying, suspended from a hook by a silk fibre, a glass thermometer which also serves as a sinker. The weights provided are in the form of riders, the two

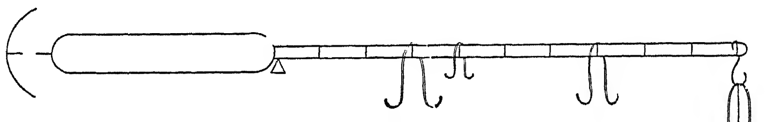
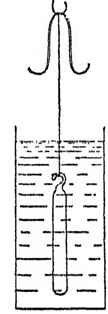


Fig. 4



largest being equal, the other two being 0·1 and 0·01 respectively of the largest weights. The hook is so adjusted that a body suspended from it is in position at the tenth mark.

The other arm of the balance carries a counterpoise with a pointed end, which point, when the balance is in equilibrium, is exactly opposite to a fiducial mark on the fixed support. Suppose now that the balance is levelled and is in equilibrium, the sinker being in air. Place the sinker in water at 15°. *It will be found that one of the heaviest weights, suspended from the hook, will restore equilibrium.* A little thought, based on a knowledge of the law of moments, should convince the student that the specific gravity of a liquid, correct to 0·001, can be read off at once from the positions of the various riders when the balance is in equilibrium, the sinker being immersed in the liquid. Thus the specific gravity of the liquid, the riders being disposed as in fig. 4, is 1·374 referred to water at 15°, and may be expressed as a density by means of Table I, p. 9.

It happens on occasion that a determination of specific gravity is called for, and that no suitable instruments are at hand. It is worth while knowing that an *accurate* result may be obtained with no more elaborate apparatus than a wooden rod, which need not be uniform, but should be uniformly graduated, and a few counterpoises of unknown weight, made of material unacted on by the liquid under test. Suppose a knitting needle, or a piece of a small triangular

file, to be fixed in the rod to form a fulcrum (fig. 5). Suspend the weights W and W_1 from the rod by loops of thread, and move W_1

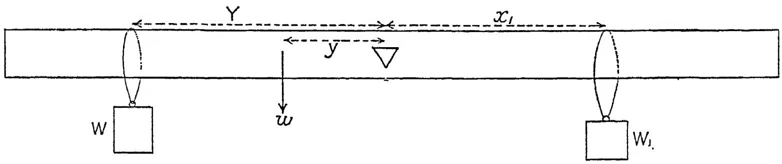


Fig. 5

until the rod is level. If w be the weight of the rod acting at a distance y from the fulcrum, we have

$$WY + wy = W_1 x_1.$$

If now, without disturbing W, we allow W_1 to hang in a beaker of water at a temperature $t°$, we have, if the point of balance be now shifted to x_2, and W_2 be the *apparent* weight of W_1,

$$WY + wy = W_2 x_2.$$

These equations give

$$W_1 x_1 = W_2 x_2, \quad \text{or} \quad \frac{W_1}{W_1 - W_2} = \frac{x_2}{x_2 - x_1}.$$

If a beaker of the liquid under test at a temperature $t_1°$ be substituted for the water and the balance point be now at x_3, we have by similar reasoning

$$W_1 x_1 = W_3 x_3 \quad \text{or} \quad \frac{W_1}{W_1 - W_3} = \frac{x_3}{x_3 - x_1}.$$

Hence

$$s_t^{t_1} = \frac{W_1 - W_3}{W_1 - W_2} = \frac{x_3 - x_1}{x_3} \cdot \frac{x_2}{x_2 - x_1},$$

and the specific gravity, which may as before be reduced to density by means of Table I, p. 9, is given accurately in terms of lengths measured along the rod.

The variation of density with temperature has been the subject of many investigations; most of the equations proposed to represent this variation (under certain specified conditions) are applicable, with great exactness, over limited ranges only. A formula has, however, been put forward recently, which gives the relation between *orthobaric* density and temperature with very considerable accuracy, over the whole range of existence of the liquid phase. It is developed thus:

We shall see later that, for unassociated liquids, the relation between surface tension and *reduced* temperature (m) is given by

$$\gamma = \gamma_0(1 - m)^n,$$

where n varies slightly from liquid to liquid, but does not deviate greatly from the value 1·2. Further, for any one such liquid the relation between surface tension and the densities of the liquid and vapour phases is

$$\gamma = C(\rho_l - \rho_v)^p,$$

where p does not deviate greatly from the value 4. Eliminating γ between these two equations, we find

$$\rho_l - \rho_v = B(1 - m)^{n/p}$$

where B stands for the p^{th} root of γ_0/C. If we assume that, at the absolute zero, the density of the super-cooled liquid is about four times the critical density, and that of the vapour is negligible, we have

$$\rho_l - \rho_v = 4\rho_c(1 - m)^{0\cdot3},$$

assuming constant values for n and for p. But it is well known that, if we take the mean of the orthobaric densities of liquid and vapour at any temperature, and plot these mean values against temperature, the result is, to a high degree of approximation, a straight line inclined at an obtuse angle to the temperature axis. That is,

$$\rho_l + \rho_v = P - Qm.$$

The condition that, at the critical point ($m = 1$) we have $\rho_l = \rho_v = \rho_c$, combined with the condition previously mentioned that for $m = 0$ we have $\rho_l = 4\rho_c$ and $\rho_v = 0$, gives us

$$\rho_l + \rho_v = 4\rho_c - 2\rho_c m.$$

Taking these equations for $(\rho_l + \rho_v)$ and $(\rho_l - \rho_v)$, eliminating ρ_v and dropping the subscript l, we find

$$\rho = 2\rho_c[(1 - m)^{0\cdot3} + (1 - 0\cdot5m)],$$

which is a reduced equation between density and temperature applicable to all unassociated substances. The equation may be tested by writing it in the form

$$\frac{\rho}{2\rho_c} - (1 - 0\cdot5m) = (1 - m)^s,$$

or

$$Y = X^s,$$

where s may or may not be equal to 0·3. A logarithmic plot of Y against X shows, in general, very good straight lines, whose slope deviates very little from the value 0·3. *But the lines do not pass through the origin.*

It follows then that

$$Y = GX^s$$

$$\rho = 2\rho_c[G(1 - m)^{0\cdot3} + (1 - 0\cdot5m)],$$

where G is a constant, whose value varies from liquid to liquid. The variation is not great, and a mean value of G is about 0·91. This general form gives very satisfactory results, but, if very close agreement with the experimental figures is necessary, the values of G and of s special to the particular liquid must be chosen.

If we put $m = 0$ in this equation, and take the value of G as 0·91, we see that the (absolute) zero density of the supercooled liquid is about $3\cdot82\rho_c$, probably a better approximation than the usual value $4\rho_c$.

Again, the equation enables us to compute a reasonably good value for the critical density if the density at any one *reduced* temperature is known. This, of course, involves a knowledge of the critical temperature. If the critical temperature is not known we may make use of Guldberg's rule, that for unassociated liquids the boiling-point under normal pressure is very approximately two-thirds of the critical temperature. Putting, therefore, $m = \frac{2}{3}$, we have

$$\rho_b = 2\rho_c[0\cdot91(1 - \tfrac{2}{3})^{0\cdot3} + (1 - 0\cdot5 \times \tfrac{2}{3})],$$

or

$$\rho_b = 2\cdot642\rho_c,$$

a general relation between the density at the normal boiling-point and the critical density.

By eliminating ρ_e instead of ρ_v a similar formula may be derived to show the march with temperature of the density of the saturated vapour.

Compressibility

We have seen that a perfect liquid is, as a matter of definition, incompressible—that is, its bulk modulus is infinite. Liquids in nature are under ordinary circumstances very slightly compressible,* and the determination of their compressibilities is, in effect, a de-

* Constantinesco, p. 222.

termination of their bulk moduli which, at any given pressure and temperature, is defined by the equation

$$\kappa = \delta p \Big/ \frac{\delta v}{v} = v\left(\frac{\partial p}{\partial v}\right)_{\mathrm{T}},^{*}$$

where δp is the additional stress (i.e. pressure per unit area) causing a *decrease* in volume δv of a substance whose initial volume is v, and $\left(\frac{\partial p}{\partial v}\right)_{\mathrm{T}}$ stands for the rate of decrease of pressure with volume under isothermal conditions (T constant). The compressibility, at any given pressure and temperature, may be defined as the reciprocal of the bulk modulus, i.e. the ratio

$$\frac{\delta v}{v} \Big/ \delta p = \frac{1}{v}\left(\frac{\partial v}{\partial p}\right)_{\mathrm{T}}.$$

Another definition of compressibility is sometimes used, namely, $\left(\frac{\partial v}{\partial p}\right)_{\mathrm{T}}$; in this case v is the volume of unit mass of the liquid. We shall here confine ourselves to a discussion of the compressibilities of liquids—those of gases and vapours will be treated later. It is clear that a complete study of the compressibility of a liquid resolves itself into the drawing of a p, v, T surface for the substance in question, so that the volume of 1 gm. of the substance is known at any pressure and temperature. The importance of this knowledge can hardly be overestimated. When we have drawn the p, v, T surface for any liquid we are in a position completely to determine its most important thermodynamic properties. In this connection the recent work of Bridgman † is pre-eminent in value, and we shall here give a discussion, as brief as may be, of his work, leaving the reader to study details of the older experiments, if he be so minded, in other books. The principles involved are simple, but it must be remembered that the experimental difficulties, when the pressures are pushed up to the order of 20,000 Kgm. per square centimetre, are very great.

The substance under test is placed in a strong chrome-vanadium steel cylinder, and the pressure is produced by the advance of a piston of known cross-section, the amount of advance of the piston

* This notation means that p is regarded as a function of v and T, and T is kept constant in finding the derivative $\frac{\partial p}{\partial v}$.

† *Proc. Amer. Acad. Sci.*, 48, 309 (1912).

giving the change in volume. It would make the story too long were we to discuss in detail the method of packing of the piston to ensure freedom from leakage, and the manner of correction for the change in volume of the cylinder, but it may be of interest to note that the pressure was measured by the change of electrical resistance of a coil of manganin. The resistance of the coil was about 100 ohms, and it was constructed of wire, seasoned under pressure, of resistance 30 ohms per metre. For high-pressure measurements this forms a very simple and convenient form of gauge. It must, of course, be calibrated, and Bridgman performed this by making, once for all, a series of measurements of the change of electrical resistance of the wire with pressure, measuring the pressure by means of a specially constructed absolute gauge. It was found that the change of resistance with pressure was so accurately linear up to pressures of 12,000 Kgm. per square centimetre that the readings could be extrapolated with confidence up to 20,000 Kgm. The changes of resistance were measured on a specially constructed Carey Foster bridge.

The whole apparatus was immersed in a thermostat, and series of pressure-volume readings were taken at different temperatures. From these readings Table II was drawn up, exhibiting the behaviour of water up to 12,500 Kgm. per square centimetre pressure, and 80° C.

A careful study of this table will show that we can extract from it data which give very complete details of the thermodynamic properties of water within the range considered. The reader is strongly recommended to work out a few of these results; by so doing he will learn in an hour or two more of the principles of thermodynamics and of the properties of water than he would gain from a week's reading of books where everything is painstakingly explained for him. The hints given below should suffice to set him going, and should he have access to Bridgman's papers, it would be well to compare his results with the curves given by Bridgman.*

(1) Calculate the compressibility $\left(\frac{\partial v}{\partial p}\right)_{T}$ or $\frac{1}{v}\left(\frac{\partial v}{\partial p}\right)_{T}$, and plot a curve between this quantity and p at any one temperature. Repeat for various temperatures.†

* Loc. cit.

† The various thermodynamical relations given in (1) to (10) will be found in treatises on thermodynamics, e.g. *Dictionary of Applied Physics*, article "Thermodynamics". Remember that T here stands for absolute temperature.

(2) Calculate the thermal dilatation $\left(\frac{\partial v}{\partial T}\right)_p$ or $\frac{1}{v}\left(\frac{\partial v}{\partial T}\right)_p$, and plot it as a function of the pressure at various temperatures.

(3) The mechanical work done by the external pressure in compressing the liquid at constant temperature is given by

$$W = \int p\left(\frac{\partial v}{\partial p}\right)_T dp$$

between given limits, and is obtained by mechanical integration (planimeter, square counting, or the like) of the curves showing the relation between p and v at constant temperature.

(4) The total heat given out, Q, during an isothermal compression is similarly derived by mechanical integration from

$$\left(\frac{\partial Q}{\partial p}\right)_T = -T\left(\frac{\partial v}{\partial T}\right)_p,$$

using the results of (2) to plot the desired curve.

(5) Knowing the mechanical work and the heat liberated in compression, we can find the difference between these, thus giving the change of internal energy along an isothermal, and can plot this against the pressure.

(6) The pressure coefficient is given by

$$\left(\frac{\partial p}{\partial T}\right)_v = -\frac{\left(\frac{\partial v}{\partial T}\right)_p}{\left(\frac{\partial v}{\partial p}\right)_T}.$$

It can thus be determined with the aid of the results of (1) and (2), and can be plotted against the pressure.

(7) The specific heat at constant pressure may be obtained by mechanical integration from the equation

$$\left(\frac{\partial C_p}{\partial p}\right)_T = -T\left(\frac{\partial^2 v}{\partial T^2}\right)_p.$$

This, of course, involves working out the second derivative from the known values of $\left(\frac{\partial v}{\partial T}\right)_p$ in the same manner as the first derivative is worked out from the original tables. Values of the specific heat as a function of the temperature at atmospheric pressure may be taken, as Bridgman took them, from the steam tables of Marks and Davis.

TABLE II

VOLUME OF ONE GRAMME OF WATER AS A FUNCTION OF PRESSURE AND TEMPERATURE.

Pressure, Kgm. per cm.²	0°.	5°.	10°.	15°.	20°.	25°.	30°	35°.	40°.	45°.	50°.	55°.	60°.	65°.	70°.	75°.	80°.
0	1·0000	·9999	1·0001	1·0007	1·0016	1·0028	1·0041	1·0057	1·0076	1·0096	1·0118	1·0143	1·0168	1·0195	1·0224	1·0255	1·0287
500	·9771	·9778	·9786	·9796	·9808	·9821	·9837	·9854	·9873	·9894	·9916	·9940	·9965	·9992	1·0020	1·0049	1·0075
1,000	·9578	·9589	·9602	·9616	·9630	·9646	·9663	·9681	·9700	·9721	·9743	·9766	·9791	·9816	·9842	·9869	·9896
1,500	·9410	·9424	·9439	·9454	·9471	·9488	·9506	·9525	·9544	·9564	·9586	·9599	·9632	·9657	·9682	·9707	·9732
2,000	·9260	·9276	·9293	·9310	·9327	·9345	·9364	·9383	·9403	·9423	·9445	·9467	·9489	·9513	·9537	·9561	·9585
2,500	·9133	·9150	·9167	·9185	·9203	·9221	·9220	·9259	·9279	·9299	·9320	·9341	·9363	·9386	·9409	·9433	·9457
3,000	·9015	·9032	·9050	·9068	·9087	·9106	·9105	·9144	·9164	·9184	·9205	·9226	·9247	·9269	·9292	·9314	·9337
3,500	·8907	·8924	·8943	·8961	·8979	·8998	·8997	·9063	·9056	·9076	·9096	·9117	·9138	·9160	·9182	·9204	·9226
4,000	·8807	·8825	·8843	·8861	·8880	·8889	·8897	·8936	·8956	·8976	·8996	·9016	·9037	·9058	·9080	·9101	·9123
4,500	·8717	·8734	·8751	·8770	·8788	·8807	·8805	·8844	·8864	·8884	·8904	·8924	·8945	·8965	·8966	·9008	·9028
5,000	·8632	·8649	·8666	·8684	·8702	·8721	·8719	·8758	·8778	·8798	·8818	·8838	·8858	·8879	·8899	·8920	·8940
5,500	·8554	·8569	·8585	·8603	·8621	·8640	·8639	·8678	·8698	·8718	·8737	·8757	·8777	·8798	·8818	·8838	·8858
6,000	·8480	·8494	·8509	·8527	·8545	·8564	·8564	·8603	·8623	·8643	·8662	·8682	·8702	·8722	·8742	·8762	·8781
6,500	·8409	·8423	·8438	·8454	·8473	·8492	·8493	·8532	·8552	·8572	·8591	·8611	·8631	·8650	·8670	·8689	·8709
7,000			·8370	·8386	·8404	·8424	·8425	·8465	·8485	·8505	·8524	·8544	·8564	·8583	·8602	·8621	·8640
7,500			·8305	·8321	·8338	·8360	·8361	·8401	·8421	·8441	·8460	·8480	·8499	·8519	·8538	·8557	·8575
8,000				·8259	·8275	·8298	·8300	·8340	·8360	·8380	·8399	·8419	·8438	·8457	·8477	·8495	·8513
8,500				·8200	·8216	·8240	·8262	·8283	·8303	·8323	·8342	·8361	·8381	·8400	·8419	·8437	·8455
9,000					·8160	·8185	·8208	·8229	·8249	·8269	·8288	·8308	·8327	·8346	·8364	·8383	·8401
9,500						·8133	·8156	·8178	·8198	·8218	·8237	·8256	·8275	·8294	·8313	·8331	·8349
10,000						·8083	·8107	·8129	·8149	·8169	·8188	·8207	·8226	·8245	·8264	·8282	·8300
10,500							·8060	·8082	·8102	·8122	·8141	·8160	·8179	·8198	·8216	·8235	·8252
11,000								·8036	·8056	·8076	·8095	·8114	·8133	·8152	·8170	·8188	·8206
11,500								·7991	·8011	·8031	·8050	·8069	·8088	·8107	·8125	·8143	·8160
12,000									·7966	·7986	·8005	·8024	·8043	·8062	·8080	·8098	·8115
12,500									·7922	·7942	·7961	·7980	·7999	·8017	·8036	·8054	·8071

(8) Knowing C_p, we can determine C_v from the equation

$$C_p - C_v = -T\frac{\left(\frac{\partial v}{\partial T}\right)_p^2}{\left(\frac{\partial v}{\partial p}\right)_T}.$$

(9) The rise in temperature accompanying an adiabatic change of pressure of 1 Kgm. per square centimetre may be deduced with the help of the equation

$$\left(\frac{\partial T}{\partial p}\right)_\phi = \frac{T}{C_p}\left(\frac{\partial v}{\partial T}\right)_p,$$

all the quantities on the right-hand side of the equation being known from the results of previous sections. ϕ refers to the entropy.

Finally (10): The difference between the adiabatic and isothermal compressibilities is given by

$$\left(\frac{\partial v}{\partial p}\right)_\phi - \left(\frac{\partial v}{\partial p}\right)_T = \frac{T}{C_p}\cdot\left(\frac{\partial v}{\partial T}\right)_p^2,$$

and may therefore be calculated.

Bridgman has added to the value of this work by making similar studies of twelve organic liquids. For details the student should consult his original papers.*

Interesting relations exist between the compressibility of a liquid and certain other of its physical constants; these we shall discuss later under other heads. Meantime we pass on to a consideration of:

Surface Tension

We may take it as an experimental fact easily deduced from the most ordinary observations that the surface of a liquid is in a state of tension and is the seat of energy. The spherical shape of small raindrops or of small globules of mercury shows that the liquid surface tends to become as small as possible in the circumstances, for a sphere is that surface which for a given content has the smallest superficial area. Again, the fact that the surface is the seat of energy is illustrated by a simple experiment suggested by Clerk Maxwell. Imagine a large jar containing a mixture of oil and water well shaken

* *Proc. Amer. Acad. Sci.*, 49, 3 (1913).

up, so that the oil is dispersed through the water in small globules. If the system be left for some time it will be found that the oil has "settled out", and it is clear that the settling-out process has involved the motion of considerable masses of matter—that is, a definite amount of work has been done. The only difference between the two states of the system is that before the settling out the surface-area of the oil-water interface was considerably greater than the area of the interface in the final state. We conclude that the surface possesses *energy*, and it will be seen shortly that an important relation exists between surface energy and surface tension.

We assume, then, that across any line of length ds drawn in the surface of a liquid there is exerted a tension γds, the direction of this tension being *normal* to the element ds and in the tangent plane to the surface. The quantity γ is called the *surface tension* of the liquid; its dimensions are clearly those of force ÷ length, and a surface tension is reckoned, in C.G.S. units, in dynes *per centimetre*, or in grammes per second per second. This tension differs from the tension in a sheet of stretched india-rubber, with which it is commonly compared, in that it is, within wide limits, independent of the area of the surface. It is constant at constant temperature, but varies with the temperature, and the calculation of its temperature coefficient is a matter of great theoretical importance. Textbook writers usually give the relation between surface tension and temperature in the form

$$\gamma = \gamma_0(1 - \alpha t),$$

a result of little value, holding good over a very limited range. The small value attaching to the formula can be shown at once, if we remember that at the critical temperature the surface tension vanishes, so that we must have $t_c = \dfrac{1}{\alpha}$. But values of t_c calculated in this way are wildly wrong, showing that the range of the formula is exceedingly restricted. It can be shown that, for liquids which do not show molecular complexity, the relation between surface tension and temperature is given by

$$\gamma = \gamma_0(1 - bt)^n,$$

where n varies from liquid to liquid, but in general has a value not differing very greatly from 1·2. This equation holds good from freezing point to critical temperature, and its accuracy may be tested

by comparing the value of t_c obtained from direct experiment with that obtained from the relation $t_c = \frac{1}{b}$. The test is shown in Table III below.

TABLE III

Substance.	n.	b.	t_c Calculated.	t_c Observed.	Difference.
			Cent. Degrees.	Cent. Degrees.	
Ether	1·248	0·005155	194	193·8	+ 0·2
Benzene	1·218	0·003472	288	288·5	− 0·5
Carbon tetrachloride	1·206	0·003553	281·5	283·1	− 1·6
Methyl formate ..	1·210	0·004695	213	214·0	− 1·0
Propyl formate ..	1·231	0·003774	265	264·9	+ 0·1
Propyl acetate ..	1·294	0·003623	276	276·2	− 0·2

We have previously referred to the relation between surface tension and surface energy. The assumption that the surface tension of a surface is equal to its surface energy (per unit area) is another common error. The "proof" given usually follows these lines. Imagine a soap film stretched over the vertical wire frame shown in fig. 6, the bar CD being movable. If the bar be pulled downwards through a distance δx, the work done against the surface forces is $2\gamma l.\delta x$ (remember that the film has two surfaces). But if λ be the energy per unit area, since the increase of surface is $2l\delta x$, the increase of superficial energy is $\lambda.2l\delta x$. Hence, equating these quantities, we have

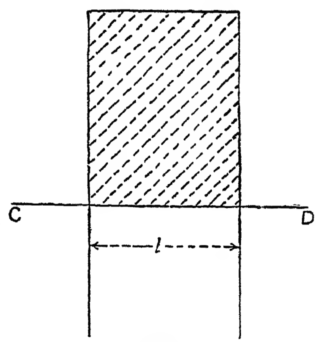


Fig. 6

$$\gamma = \lambda.$$

But this argument overlooks the important fact that surface tension diminishes with increasing temperature. Hence it follows from thermodynamic principles that, in order to stretch the film isothermally, heat must flow into the film to keep its temperature constant,

and this heat goes to increase the surface energy. A simple thermodynamic argument shows that the relation between γ and λ is given by

$$\gamma = \lambda + T\frac{\partial \gamma}{\partial T},$$

where T stands for absolute temperature, and only if the temperature coefficient of surface tension were zero would the simpler equation hold.

Since we know the relation between surface tension and temperature for unassociated substances we can easily work out, by substituting for γ and $\frac{\partial \gamma}{\partial T}$ in the equation just given, the relation between surface energy and temperature. This relation is shown in fig. 7, the dotted curve showing the variation of surface tension, the full curve the variation of surface energy with temperature. The two curves intersect each other and the axis of temperature at the critical temperature, showing that at that point both surface tension and surface energy vanish. But for lower temperatures the two quantities are in general very different in numerical magnitude, surface energy increasing much faster than surface tension with falling temperature. This important fact should carefully be borne in mind.

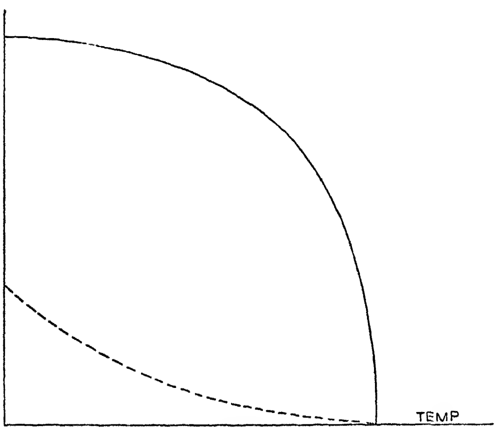


Fig. 7

Many relations, empirical and otherwise, have been suggested connecting surface tension with other physical constants. Thus, Macleod * has recently found that for any one liquid at different temperatures

$$\gamma = C(\rho_l - \rho_v)^4,$$

* *Trans. Faraday Soc.*, 1923. The present writer has also shown (*Trans Faraday Soc.*, July, 1923) that the constant C may be expressed in the form $C = \Delta T_c / M^{\frac{2}{3}}\rho_c^{\frac{10}{3}}$, where M is the molecular weight, ρ_c the critical density, and Δ a constant independent of the nature of the liquid.

where C is a constant independent of the temperature, ρ_l the density of the liquid, and ρ_v that of the saturated vapour of the liquid.

We should naturally expect surface tension and compressibility (β) to stand in intimate relation, and experiment shows that liquids of high compressibility have low surface tensions and conversely. Richards and Matthews * have examined the quantitative relation between these two constants, and find that, for a large number of unassociated substances, the product $\gamma\beta^{\frac{4}{3}}$ is a constant quantity.

A most important equation connecting surface tension, density, and temperature, is that proposed by Eötvös,†

$$\gamma\left(\frac{M}{\rho}\right)^{\frac{2}{3}} = K(T_c - T - \delta),$$

where M is the molecular weight of the liquid, ρ its density, and δ and K are constants for unassociated liquids, δ being about 6 and K 2·12. The equation shows that a knowledge of the temperature variation of γ enables us to calculate the molecular weight of the liquid under examination, and hence to determine whether its molecules are or are not associated.

In recent years this test of association has been slightly altered. Instead of examining the variation of $\gamma\left(\frac{M}{\rho}\right)^{\frac{2}{3}}$ with temperature, the variation of $\lambda\left(\frac{M}{\rho}\right)^{\frac{2}{3}}$ has been studied, where, as we have seen,

$$\lambda = \gamma - T\frac{\partial\gamma}{\partial T}.$$

Bennett and Mitchell ‡ have shown that for unassociated liquids this quantity, which we may call the *total* molecular surface energy, is constant over a fairly wide range of temperature, and have used this constancy as a test of non-association.

We now turn to the discussion of a problem of fundamental importance—that of the relation between the pressure-excess (positive or negative) on one side of a curved surface and the tension in the surface. It is fairly clear that the pressure just inside a curved surface such as that of a spherical bubble is greater than the pressure just outside the surface, and the manner in which pressure-excess is connected with surface tension may be calculated as follows.

* *Zeit. Phys. Chem.*, **61**, 49 (1908).
† See Nernst, *Theoretical Chemistry*, p. 270 (1904).
‡ *Zeit. Phys. Chem.*, **84**, 475 (1913).

Imagine a cylindrical surface whose axis is perpendicular to the plane of the paper, part of the trace of the surface by the plane of the paper being the curve AB (fig. 8). Consider the equilibrium of a portion of this cylindrical surface of unit length perpendicular

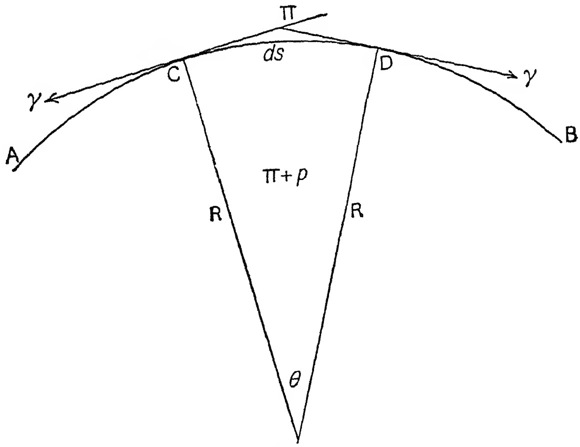


Fig. 8

to the plane of the paper, and of length ds in the plane of the paper. If Π and $\Pi + p$ be the pressures at CD on the two sides of the cylinder, we have, resolving normally,

$$2\gamma \sin\frac{\theta}{2} + \Pi ds = (\Pi + p)ds,$$

when θ is the radian measure of the angle indicated in fig. 8, or, since θ is small,

$$\gamma\theta = pds.$$

But $\theta = \dfrac{ds}{R}$ where R is the radius of curvature at C, and therefore

$$p = \frac{\gamma}{R}.$$

If the surface is one of double curvature, the effects are additive and we have

$$p = \gamma\left(\frac{1}{R_1} + \frac{1}{R_2}\right),$$

where R_1 and R_2 are the principal radii of curvature at the point in question. Thus for a spherical drop, or a spherical air-bubble in a liquid, we have

$$p = \frac{2\gamma}{R},$$

where R is the radius of the drop or bubble. For a spherical *soap-bubble*, which has two surfaces, we should have

$$p = \frac{4\gamma}{R}.$$

The use of the pressure-excess equation, combined with a knowledge of the fact that a liquid meets a solid at a definite angle called the contact-angle, will suffice to solve many important surface-tension problems. Thus the rise or fall of liquids in capillary tubes is readily explained. Water, for example, meets glass at a zero contact

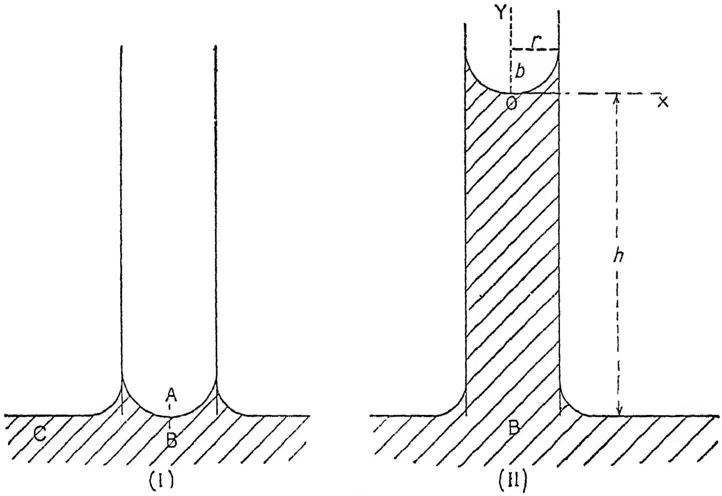


Fig. 9

angle; hence the surface of water in a capillary tube must be sharply curved, and the narrower the tube the sharper must be the curvature in order that the liquid may meet the glass at the proper angle. The state of affairs shown in fig. 9 (i) is impossible; for the pressure at A being atmospheric, the pressure at B must be less than atmospheric by $\frac{2\gamma}{R}$, where R is the radius of curvature of the meniscus at B. But the pressure at C in a liquid at rest must be equal to that at B, and the pressure at C is clearly atmospheric. Hence the liquid must rise in the tube until the additional pressure due to the head h just brings the pressure at B up to atmospheric value. We must have therefore

$$\frac{2\gamma}{R} = g\rho h.$$

If the tube be very narrow—and the criterion of narrowness that is $\frac{r}{h}$ shall be small compared with unity—the meniscus will be a segment of a sphere, and the contact angle being zero we may put $R = r$, the radius of the tube, giving the well-known equation

$$\gamma = \tfrac{1}{2}g\rho rh.$$

If $\frac{r}{h}$, though small, be not negligible compared with unity the meniscus will be flattened; a very close approximation to the truth may be obtained by treating the meridional curve as the outline of a semi-ellipse. Suppose the semi-axes of the ellipse to be r and b (fig. 9 ii). If we take the contact angle as zero, there will be an upward pull of $2\pi r\gamma$ on the liquid in the tube all round the line of contact of the liquid with the glass. Equating this to the weight of liquid raised (including the weight of that in the meniscus) we have

$$2\pi r\gamma = \pi r^2 h\rho g + \tfrac{1}{3}\pi r^2 b\rho g,$$

or

$$2a^2 = rh + \tfrac{1}{3}rb,$$

if for brevity we write a^2 for $\frac{\gamma}{g\rho}$. But if R be the radius of curvature at O, we have *accurately*

$$\text{Pressure-excess} = g\rho h = \frac{2\gamma}{R},$$

or

$$2a^2 = Rh.$$

Now R, the radius of curvature at the end of the semi-axis minor of an ellipse, is equal to $\frac{r^2}{b}$. Hence

$$b = \frac{r^2 h}{2a^2},$$

and therefore

$$2a^2 = rh + \tfrac{1}{3}r.\frac{r^2 h}{2a^2},$$

or

$$12a^4 - 6rha^2 - r^3 h = 0.$$

Solving this as a quadratic in a^2 and expanding the surd, we obtain

$$2a^2 = rh\left(1 + \tfrac{1}{3}\frac{r}{h} - 0{\cdot}1111\frac{r^2}{h^2} + 0{\cdot}0741\frac{r^3}{h^3}\ldots\right).$$

In all practical cases $\frac{r}{h}$ is small compared with unity, and the above

equation gives values for a^2 (and therefore for γ) in close numerical agreement with those obtained from the equation

$$2a^2 = rh\left(1 + \frac{1}{3}\frac{r}{h} - 0{\cdot}1288\frac{r^2}{h^2} + 0{\cdot}1312\frac{r^3}{h^3}\right),$$

obtained by the late Lord Rayleigh * as the result of a rather complex and difficult analysis.

The problem of the measurement of interfacial tensions has recently assumed great technological importance, mainly on account of the rapid development of colloid chemistry and physics. Tanning, dyeing, dairy chemistry, the chemistry of paints, oils, and varnishes, of gums and of gelatine are all concerned deeply with the properties of colloidal systems in which one phase is dispersed in very small particles through the substance of another phase. There is consequently a relatively great extent of surface developed between the two phases, and the interfacial tension at the surface of separation of the phases may play an important, not to say decisive, part in determining the behaviour of the system.

This tension may be measured by a modification of the capillary tube experiment described above. For example, the tension at a benzene-water interface has been measured by surrounding the capillary with a wider tube, and filling with benzene the space previously occupied by air.

Exact determinations can conveniently be made by the drop-weight method, wherein a drop of liquid is formed at the end of and detached *slowly* from a vertical thick-walled capillary tube immersed in the second (and lighter) liquid. The method can, of course, be used to determine liquid-air tensions. So many erroneous statements have been made concerning the practice and theory of this method that it is worth while considering it in some detail. For example, a common practice in physico-chemical works is to equate the weight of the detached drop to $2\pi r\gamma$, a procedure which, but for the fact that the drop-weight method is often used as a comparative one, would give results about 100 per cent in error. Those, again, who are alive to the error of writing

$$mg = 2\pi r\gamma$$

not infrequently tell us that the constant 2π must be replaced by another constant of value 3·8, for no very apparent or adequate reason. Let us then investigate the problem as exactly as may be

* *Proc. Roy. Soc.* **92** (A), 184 (1915).

in an elementary manner, and see if some justification exists for this procedure. Suppose for the moment that the drop is formed in air. If we assume that the drop is *cylindrical* at the level AB (fig. 10), then, Π being the atmospheric pressure, the pressure at any point in the plane AB is $\Pi + \frac{\gamma}{r}$. Consequently, resolving vertically for the forces acting on the portion of the drop below AB, we have

$$mg + \left(\Pi + \frac{\gamma}{r}\right)\pi r^2 = 2\pi r\gamma + \Pi.\pi r^2,$$

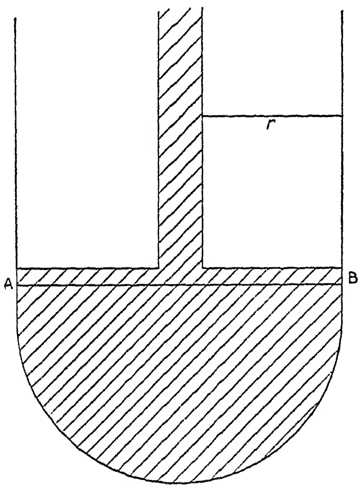


Fig. 10

leading to

$$mg = \pi r\gamma,$$

exactly half the value of the weight of a drop as given by most of the textbooks.

But the detachment of a drop is essentially a dynamical phenomenon, and no statical treatment can be complete. We can, however, obtain some assistance from the theory of dimensions. Assume that the mass of a detached drop depends on the surface tension and the density of the liquid, the radius of the tube, and the acceleration due to gravity. We may thus write

$$m = K\gamma^x g^y \rho^z r^w.$$

Dimensionally

$$[M] = [MT^{-2}]^x[LT^{-2}]^y[ML^{-3}]^z[L]^w,$$

leading to

$$x + z = 1, \; x + y = 0, \; y - 3z + w = 0.$$

Solving for w, y, and z in terms of x we find

$$m = K \cdot \frac{\gamma r}{g}\left(\frac{\gamma}{g\rho r^2}\right)^{x-1},$$

or

$$m = \frac{\gamma r}{g} F\left(\frac{\gamma}{g\rho r^2}\right),$$

where F is some arbitrary function of the variable $\frac{\gamma}{g\rho r^2}$. The late Lord Rayleigh determined the weight of drops of water let fall slowly from tubes of various external diameters. Knowing the surface tension of water, he was enabled to tabulate the variation of the function F with that of the independent variable $\frac{\gamma}{g\rho r^2}$; for, as we see, the function F is given by

$$F = \frac{mg}{\gamma r}.$$

In this way the following table was drawn up.

TABLE IV

$\frac{\gamma}{g\rho r^2}$.	$F\left(\equiv \frac{mg}{\gamma r}\right)$.
2·58	4·13
1·16	3·97
0·708	3·80
0·441	3·73
0·277	3·78
0·220	3·90
0·169	4·06

It will be seen that for a considerable variation of the variable $\frac{\gamma}{g\rho r^2}$ —and this means a considerable variation of r—the function F does not fluctuate seriously, and for most purposes it is permissible to assume that F is constant and equal to 3·8. Hence the reason for the equation

$$mg = 3{\cdot}8r\gamma.$$

The argument for interfacial tensions follows identical lines, and the reader should have no difficulty in working it out for himself, remembering that the drop of density ρ, say, is now supposed to be pendent in a lighter liquid of density ρ_1.

If we assume that the liquids with which we are dealing obey the power law for the variation of surface tension with temperature, we have

$$\gamma = \gamma_0(1 - m)^n,$$

where, for convenience, we express temperatures in the reduced form. The total surface energy λ is given by

$$\lambda = \gamma - m\frac{\partial \gamma}{\partial m},$$

and, with this form for γ, is

$$\lambda = \gamma_0(1 - m)^{n-1}\{1 + (n - 1)m\},$$

and we see that, contrary to some statements, there is no indication of a maximum value for λ, the march of λ with temperature following the curve shown in fig. 7.

We have seen that a reduced equation may be developed between orthobaric density and temperature which, in its simplest form, may be written

$$\rho = 2\rho_c[(1 - m)^{0\cdot 3} + 1 - 0\cdot 5m].$$

It follows then that free molecular surface energy (e) defined as $\gamma(M/\rho)^{\frac{2}{3}}$ and total molecular surface energy (E) defined as $\lambda(M/\rho)^{\frac{2}{3}}$, have their variation with temperature at once determined on substituting in these expressions the appropriate expressions for γ, λ and ρ.

The deduction of the equations showing how e and E vary with the temperature is left to the reader, but it may be noted that e is not a linear function of the temperature, nor is E independent of the temperature, although the variation at fairly low temperatures is very small, and the assumption of constancy over ordinary ranges of temperature need lead to no serious error. Nevertheless it is worthy of note that, considering the whole range $m = 0$ to $m = 1$, the quantity E rises very slowly to a not very pronounced maximum at a temperature about $\frac{4}{5}$ of the critical value, thereafter falling rapidly to zero at the critical point. It is interesting to see that this slight maximum is shown in the experimental figures, but was overlooked, as workers in the subject were looking rather for constancy than variation with temperature.

Some time ago Katayama remarked that very considerable simplification resulted if the difference of the liquid and vapour densities were substituted for the liquid density in the definitions of E and e. We thus have

$$e = \gamma\left(\frac{M}{\rho_e - \rho_v}\right)^{\frac{2}{3}} \quad \text{and} \quad E = \lambda\left(\frac{M}{\rho_e - \rho_v}\right)^{\frac{2}{3}},$$

and Katayama points out that, in these circumstances, e and E are linear functions of the temperature given by

$$e = e_0(1 - m), \quad E = E_0(1 + 0{\cdot}2m).$$

As the reader may easily convince himself, these results depend on the power law being followed with n equal to 1·2, and Macleod's law being obeyed with the index equal to 4.

If we write this latter law in the more general form

$$\gamma = C(\rho_e - \rho_v)^p,$$

and do not assume any special value for n in the expression for the power law, we readily find that $E = E_0(1 - m)^x\{1 + (n - 1)m\}$, where for brevity x is written for $(n - 1 - 2n/3p)$. If $n = 1{\cdot}2$ and $p = 4$, we have $x = 0$, and Katayama's value for E results. In no instances that we have examined is this exactly true. The index x is small but positive, and the result is that E climbs by an almost linear ascent to a definite maximum, thereafter falling very rapidly to zero at the critical point—behaviour much more consonant with our usual conception of surface energy than that given by Katayama's equation, which gives E its highest value at the critical point.

To establish these results is not difficult. If we have, for a substance whose critical temperature is known, a series of values of surface tensions determined over a wide temperature range, a logarithmic plot of $(1 - m)$ and γ serves to test the power law, and to determine the value of n where the power law is followed. The values of λ, e and E at different temperatures may then readily be computed. E_0, the zero value of the total molecular surface energy, is readily deduced, and it may be remarked that this quantity varies in very interesting and regular fashion with variation in chemical constitution.

The quantity $M\gamma^{\frac{1}{4}}/(\rho_e - \rho_v)$ (where M stands for molecular weight) has been named the *parachor.* It provides us with a number which measures the molecular volume of a liquid *at a temperature at which the surface tension is unity*, and therefore gives a most valuable means of comparing molecular volumes under corresponding conditions.

Viscosity

We have seen that a perfect fluid is one in which tangential stresses do not exist, whether the fluid be at rest, or whether its different portions be in motion relative to each other. Such stresses do,

however, appear in all known fluids when relative motion exists, and the fluid may be looked upon as yielding under the stress, different fluids yielding at very different rates.

The most obvious effect of the existence of such tangential stresses between different parts of the fluid is the tendency to damp out relative motion. Thus, if we have a layer of liquid flowing over a plane solid surface, the flow taking place in parallel horizontal layers, the layer of liquid in contact with the surface will be at rest, and there will be a steady increase, with increase of height above the solid surface, in the horizontal velocity of the successive layers. Considering the surface of separation between any two layers, the tangential stress existing there will tend to retard the faster moving upper layer, and to accelerate the slower moving lower layer. The magnitude of the tangential stress may be written down if we assume, following Newton, that the tangential stress is proportional to the velocity *gradient*, so that, if the horizontal velocity is v at a vertical distance y from the fixed surface, we have

$$\mathrm{S} \propto \frac{dv}{dy}, \text{ i.e. equals } \mu\frac{dv}{dy},$$

where μ is a constant called the coefficient of viscosity of the fluid. If $\frac{dv}{dy}$ is unity, then $\mathrm{S} = \mu$. Hence we are led to Maxwell's well-known definition of μ: "The viscosity of a substance is measured by the tangential force on unit area of either of two horizontal planes of indefinite extent at unit distance apart, one of which is fixed, while the other moves with unit velocity, the space between being filled with the viscous substance".

The dimensions of μ are those of stress divided by velocity gradient; this works out to

$$[\mu] = [\mathrm{ML}^{-1}\mathrm{T}^{-1}],$$

so that a coefficient of viscosity in C.G.S. units is correctly given as x gm. per centimetre per second.

If in fig. 1 (*c*), p. 4, we put Aa = dx, AD = dy, we see that the rigidity modulus (N) is given by

$$\mathrm{S} = \mathrm{N}\frac{dx}{dy}.$$

Comparing this with $\mathrm{S} = \mu\frac{dv}{dy}$, it is clear that the dimensions of

viscosity differ from those of rigidity by the time unit, in the same way as the dimensions of length differ from those of velocity. In fact, the rigidity modulus of a solid determines the *amount* of the strain set up by a given tangential stress, and the viscosity modulus of a fluid determines the *rate* at which the fluid yields to the stress.

The fall of a sphere through a viscous fluid aptly illustrates several interesting physical phenomena; we shall therefore study the problem in some little detail. Considerable assistance is given by an application of the theory of dimensions. Suppose that the resistance (R) experienced by the sphere depends on its radius (a), its velocity (v), and the density (ρ) and viscosity (μ) of the surrounding fluid, we then have

$$R = ka^x \rho^y \mu^z v^w,$$

and as the right-hand side must have the dimensions of a force, we obtain, by equating the exponents of M, L, and T,

$$x = w, \ y = w - 1, \ z = 2 - w,$$

so that

$$R = ka^w \rho^{w-1} \mu^{2-w} v^w$$
$$= k\left(\frac{va\rho}{\mu}\right)^w \cdot \frac{\mu^2}{\rho}.$$

For low velocities we may assume that the resistance is proportional to the velocity. Putting therefore $w = 1$, we find

$$R = k\mu av.$$

A more complex analysis, originally given by Stokes,* shows that

$$R = 6\pi\mu av.$$

If, however, we assume that for high velocities the resistance varies as the square of the velocity, we have, putting $w = 2$,

$$R = kv^2a^2\rho,$$

and viscosity does not enter into the question—energy is expended, not in overcoming viscous resistance, but in producing turbulent motion in the liquid.

Returning to the problem of low velocities, let us write down the equation of motion of a sphere falling vertically through an infinite ocean of fluid. The forces acting are—the weight (W) of the sphere

* Lamb, *Hydrodynamics*, p. 532 (1895).

(downwards), the resistance (R), and the buoyancy (B) of the displaced fluid (upwards). This gives

$$W - (B + R) = mf,$$

where m is the mass and f the downward acceleration of the sphere. But as the velocity of the sphere increases, R increases *pari passu*, so that the acceleration steadily diminishes, until when R has increased to such an extent that

$$W - B = R,$$

f becomes zero, and the sphere henceforward falls with a constant velocity known as the "terminal velocity". Calling this velocity V, the density and radius of the sphere ρ and a respectively, and the density of the fluid ρ_0, we have

$$\tfrac{4}{3}\pi a^3 g(\rho - \rho_0) = 6\pi\mu a V,$$

leading to

$$\mu = \frac{2}{9} \cdot \frac{(\rho - \rho_0)ga^2}{V}.$$

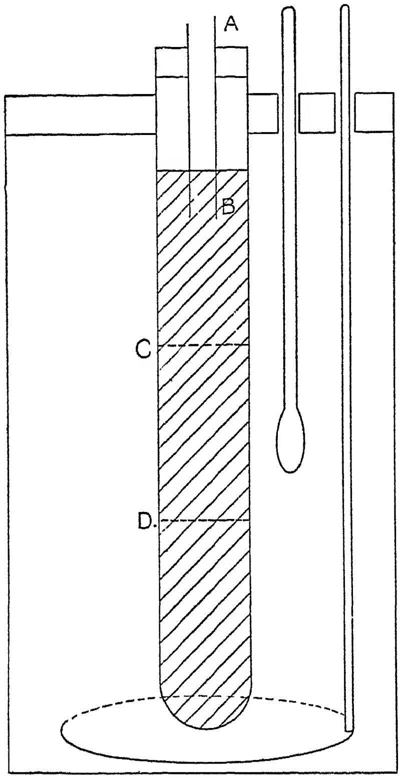


Fig. 11

Clearly, measurement of the terminal velocity V enables us to determine the viscosity of a liquid. The method is peculiarly suited for the measurement of the viscosities of very viscous liquids such as heavy oils or syrups, and has been much used of late years. The simple apparatus required is shown in fig. 11. The outer cylinder represents a thermostat; the inner cylinder contains the liquid under experiment.

The sphere—steel ball bearings 0·15 cm. in diameter are suitable for liquids having viscosities comparable with that of castor oil—is dropped centrally through the tube AB, and its velocity is measured over the surface CD, which represents one-third of the total depth of the liquid.

Two important corrections are necessary—one for "wall-effect", one for "end-effect"; for it must be remembered that the simple theory given above applies only to slow motion through an *infinite ocean* of fluid.

These corrections have been investigated by Ladenburg,* who has shown that in order to correct for wall-effect we must write

$$V\left(1 + 2{\cdot}4\frac{a}{R}\right) = V_\infty,$$

where V is the observed velocity, V_∞ the corresponding velocity in an infinite medium, and $\frac{a}{R}$ the ratio of the radius of the sphere to that of the cylinder containing the liquid.

Similarly for the end-effect

$$V\left(1 + 3{\cdot}3\frac{a}{h}\right) = V_\infty,$$

where h represents the total height of the liquid which is supposed to be divided into three equal portions, V representing the mean velocity over the middle third. Introducing these corrections into Stokes's formula, we obtain

$$\mu = \frac{2}{9}\cdot\frac{(\rho - \rho_0)ga^2}{V\left(1 + 2{\cdot}4\frac{a}{R}\right)\left(1 + 3{\cdot}3\frac{a}{h}\right)}.$$

The method has been much used during the war period for the measurement of μ for liquids of high viscosity, and is fully described in a paper by Gibson and Jacobs.†

A commercial viscometer has recently come into use, which is of exceedingly simple type, and gives fairly reliable results. A steel ball $\frac{3}{4}$ in. in diameter is placed inside a hemispherical steel cup of slightly larger dimensions. The cup carries on its internal surface three small projections in length about 0·002 in. A little of the oil under examination is poured into the cup, and the ball placed in position inside the cup. The ball is pressed down on to a table, the cup being uppermost, and at a given instant cup and ball are lifted clear of the table. The time taken for the ball to detach itself is measured, and this gives a measure of the viscosity of the oil.‡

* *Ann. der Physik* (IV), **23**, 9 and 447 (1907).

† *Jour. Chem. Soc.*, **117**, 473 (1920).

‡ For fuller description see Chapter III, p. 119.

Comparative measurements only can be made, and the instrument must be standardized by means of a liquid of known viscosity.

Viscometers for use with ordinary liquids usually depend on measurements of the flow of a liquid through a horizontal or vertical capillary tube. The solution of the problem for a horizontal capillary affords an interesting application of the general equations of hydrodynamics, and we shall attack the problem from that side. The reader may or may not be able to follow the arguments by which these equations are established—he may study them at leisure in the treatises of Lamb, of Bassett, or of Webster—what *is* important is that he should see clearly their physical significance and obtain practice in handling them. This is best done by a careful study of one or two of their applications.

The equations of motion of an incompressible fluid are: *

$$\rho\frac{\mathrm{D}u}{\mathrm{D}t} = \rho \mathrm{X} - \frac{\partial p}{\partial x} + \mu\nabla^2 u, \quad \ldots\ldots\ldots\ldots(\alpha)$$

$$\rho\frac{\mathrm{D}v}{\mathrm{D}t} = \rho \mathrm{Y} - \frac{\partial p}{\partial y} + \mu\nabla^2 v, \quad \ldots\ldots\ldots\ldots(\beta)$$

$$\rho\frac{\mathrm{D}w}{\mathrm{D}t} = \rho \mathrm{Z} - \frac{\partial p}{\partial z} + \mu\nabla^2 w, \quad \ldots\ldots\ldots\ldots(\gamma)$$

where
$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z},$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2},$$

p is the pressure at any point, X, Y, Z the components of the external force per unit mass, u, v, w the velocity components.

To apply these equations to the steady flow of a liquid through a horizontal capillary tube, we take the axis of the tube as z-axis, and assume the flow everywhere parallel to this axis. Then $u = v = 0$, and from (α) and (β) we have

$$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial y} = 0,$$

so that the mean pressure over any section of the tube is uniform.

* See Chapter II, p. 83.

Also from the equation of continuity,

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0,$$

we have
$$\frac{\partial w}{\partial z}=0.$$

Hence (α) and (β) vanish, and (γ) becomes, assuming no extraneous forces,

$$\rho\frac{\partial w}{\partial t}+\frac{\partial p}{\partial z}-\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)=0.$$

Since w varies only with t, x, and y, and p only with z, then $\frac{\partial p}{\partial z}$ = constant = c (say), and therefore

$$\rho\frac{\partial w}{\partial t}-\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)=-c.$$

We can now obtain the equation known as Poiseuille's equation, for if the motion is unaccelerated $\frac{\partial w}{\partial t}=0$, and

$$\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)=c.$$

Transforming to polar co-ordinates,*

$$\mu\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right)=c,$$

or, w being independent of θ,

$$\mu\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}\right)=c.$$

This may be written

$$\frac{\mu}{r}\left(r\frac{\partial^2 w}{\partial r^2}+\frac{\partial w}{\partial r}\right)=\frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right)=c.$$

Integrating, we have

$$r\frac{\partial w}{\partial r}=\frac{c}{\mu}\frac{r^2}{2}+A;$$

* See p. 52 *et seq.*

and integrating a second time

$$w = \tfrac{1}{4} \cdot \frac{c}{\mu} \cdot r^2 + A \log r + B,$$

where A and B are constants of integration. When $r = 0$ (on the axis of the tube), w is finite. Consequently A must be zero, and

$$w = \tfrac{1}{4} \cdot \frac{c}{\mu} \cdot r^2 + B.$$

Also, if there is no slipping of the liquid at the walls of the tube, when $r = a$, $w = 0$, and consequently

$$w = -\tfrac{1}{4} \cdot \frac{c}{\mu}(a^2 - r^2).$$

If V is the *volume* of liquid which escapes from the tube in a time T the volume issuing in unit time is given by

$$\frac{V}{T} = \int_0^a w . 2\pi r dr$$

$$= -\frac{2\pi}{4\mu} \cdot c \int_0^a (a^2 - r^2) r dr = -\frac{\pi}{8} \cdot \frac{a^4}{\mu} \cdot c.$$

If p_1 and p_2 are the pressures at the entrance to and exit from the pipe, then remembering that $\frac{\partial p}{\partial z}$ stands for the rate of *increase* of p with z, we have

$$c = \frac{\partial p}{\partial z} = -\frac{p_1 - p_2}{l},$$

where l is the length of the pipe. Hence

$$\mu = \frac{\pi}{8} \cdot \frac{a^4 T}{V} \cdot \frac{p_1 - p_2}{l}.$$

If the liquid is supplied to the tube under a constant head h, and escapes into the air at a low velocity, we have

$$\mu = \frac{\pi}{8} \cdot \frac{a^4 T}{V} \cdot \frac{g\rho h}{l}.$$

This equation is known as Poiseuille's equation. All the quantities on the right-hand side may be determined experimentally, and hence μ may be evaluated.

Comparative measurements by this method are usually made using Ostwald's viscometer (fig. 12). The bulb C is filled with the liquid under examination, which is then drawn up by suction until it fills the bulb D. The pressure is then released, and the time of transit between the marks A and B observed. The pressure head is varying throughout the fall, and clearly we cannot apply Poiseuille's equation as it stands. But, noting that for liquids of equal densities and different viscosities the times will be proportional to the viscosities, and that for liquids of equal viscosities and different densities the times will be inversely proportional to the densities, we have in general,

$$t = \mathrm{K}\frac{\mu}{\rho} \quad \text{or} \quad \mu = c\rho t,$$

where c is a constant for the apparatus to be determined by using a liquid of known viscosity.

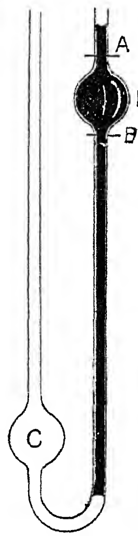


Fig. 12

A viscometer of dimensions suitable for the determination of the viscosity of water is not suited for use with heavy oils. But if we have a series A, B, C, . . . of viscometers of gradually increasing bore, calibrate A by using water, and then use the most viscous liquid suitable for A in order to calibrate B, continuing this process as far as necessary, we are provided with a chain of viscometers which can be used over a very wide range.

Since the viscosity of liquids decreases rapidly with increase of temperature, it is of vital necessity that the apparatus be enclosed in some form of thermostat and that the temperature of experiment be taken and *recorded*. This rapid change of viscosity with temperature makes it very difficult to obtain relations between the viscosities of different chemically related substances, as it is by no means easy to settle the temperature of comparison. It has been found, however, that consistent results may be obtained if viscosities are compared at *temperatures of equal slope*—that is at temperatures for which $\frac{d\mu}{dt}$ is the same. Using this standard it has been shown, for example, that the molecular viscosities * of a homologous series increase by a constant amount for each addition of CH_2.

* The molecular viscosity of a liquid is defined as $\mu(\mathrm{M}v)^{\frac{2}{3}}$, where M is the molecular weight and v the specific volume of the fluid concerned.

There are many important practical problems which depend for their solution on a knowledge of friction in fluids. The viscosities of mixtures of liquids, the viscosities of gases, the theory of lubrication, the discussion of turbulent motion, to mention but a few, present important and most interesting aspects. These matters are fully discussed in Chapter III.

One interesting problem may be mentioned in passing—the suspension of clouds in air, where we have the apparent paradox of a fluid of specific gravity unity suspended in a fluid of specific gravity 0·0013. The paradox is cleared up by an application of Stokes' formula,

$$V = \frac{2}{9}\frac{(\rho - \rho_0)ga^2}{\mu}.$$

Taking the viscosity of air as 0·00017 in C.G.S. units, the reader is recommended to calculate the terminal velocities of spheres of water, say 0·1, 0·01, . . . cm. in radius. The terminal velocities of minute drops will be found to be surprisingly small.

The kinetic theory of liquid viscosity has not received a great deal of serious attention, and formulæ developed to show, for example, the dependence of liquid viscosity on temperature have usually a purely empirical basis. Of these, one proposed by Porter may be specially noted. Suppose that the variation of viscosity with temperature has been experimentally investigated for two liquids. Take a temperature T_1 at which one liquid has a viscosity η_1. Find the temperature T_2 at which the second liquid has the viscosity η_1. Repeat for different values of T_1. Then T_1/T_2 is a linear function of T_1.

Recently Andrade has put forward a kinetic theory in which he assumes "that the viscosity is due to a communication of momentum from layer to layer, as in Maxwell's theory of gaseous viscosity, but that this communication of momentum is not effected to any appreciable extent by a movement of the equilibrium position of molecules from one layer to another, but by a temporary union at the periphery of molecules in adjacent layers, due to their large amplitudes of vibration."

This assumption leads to a formula connecting viscosity and temperature of the form

$$\eta = Ae^{\frac{c}{T}},$$

a formula which had been put forward previously on an empirical

basis. Porter's relation, as the reader may verify, follows at once from this equation.

In the deduction of this equation, variation of volume with temperature has been neglected and, taking this factor into account, Andrade deduces a second formula,

$$\eta v^{\frac{1}{3}} = Ae^{\frac{c}{vT}},$$

where v is the specific volume. For a great many organic liquids this formula gives a very good fit, though, as was to be expected, water and the tertiary alcohols show abnormalities.

Equations of State

Much labour has been expended on the problem of devising equations which shall represent accurately the pressure-volume-temperature relations of a substance in its liquid and its gaseous phases. It may be said at once that it is impossible to devise an equation which shall be accurate over such a range without being impossibly cumbrous. Nevertheless the simpler equations have, as we shall see, considerable value in giving a fairly adequate representation of the general behaviour of a homogeneous fluid.

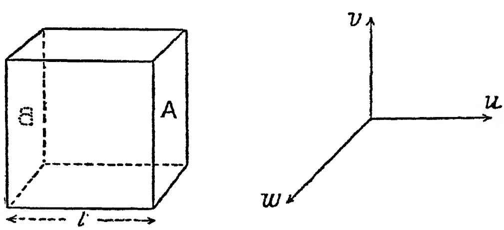


Fig. 13

A very simple type of such a fluid is a gas, considered as an assemblage of material points which are in rapid—and random—motion, and which do not exert any attractive forces on each other. Consider a given quantity of such a gas enclosed in a cube of side l, and let the component velocities of any one particle be u, v, w as shown (fig. 13).

The pressures on the faces of the cube are due to the impacts of the particles thereon. At any impact on, say, the face A the velocity component normal to that face will be reversed, and the change of momentum of the particle consequent on the rebound will be

$$mu - (-mu) = 2mu.$$

The time of travel from A to B and back is $\frac{2l}{u}$ sec.; the frequency of the impacts on the face A is $\frac{u}{2l}$; hence for any one particle the change of momentum at A per second will be

$$2mu \times \frac{u}{2l} = \frac{mu^2}{l},$$

and similarly for the other particles.

The *force* on the face A due to molecular bombardment is, therefore, $\frac{m}{l}\Sigma u^2$.

If p is the *pressure* on the face A,

$$p = \frac{1}{l^2} \cdot \frac{m}{l}\Sigma u^2 = \frac{m}{l^3}\Sigma u^2 = \frac{m}{V}\Sigma u^2,$$

where V is the volume of the cube. Similarly, for the pressures on the faces perpendicular to A, we have the expressions

$$\frac{m}{V}\Sigma v^2, \quad \frac{m}{V}\Sigma w^2.$$

But these pressures are equal, and therefore

$$p = \frac{m}{V}\Sigma u^2 = \frac{m}{V}\Sigma v^2 = \frac{m}{V}\Sigma w^2$$

$$\tfrac{1}{3}\frac{m}{V}\Sigma(u^2 + v^2 + w^2) = \tfrac{1}{3} \cdot \frac{m}{V}\Sigma U^2,$$

where $$U^2 = u^2 + v^2 + w^2.$$

Now let us define a mean velocity $\overline{U}$ by the relation

$$\overline{U}^2 = \Sigma\frac{U^2}{N},$$

N being the total number of particles in the cube. We then have

$$pV = \tfrac{1}{3}mN\overline{U}^2,$$

and mN being the mass of the gas, we have, if ρ be its density,

$$\rho = \frac{mN}{V}, \quad \text{and} \quad p = \tfrac{1}{3}\rho\overline{U}^2.$$

Hence Boyle's Law.

If we assume that $\overline{U}^2$ is proportional to the absolute temperature,

we have Charles's law, and can write as the characteristic equation of our " perfect " gas

$$pV = RT.$$

If V stands for the volume of unit mass of the gas, R will be different for different substances. A simple deduction from our fundamental equation shows, however, that the gramme-molecular volume* is the same for all gases. Consider two different gases for which

$$p_1 v_1 = \tfrac{1}{3} m_1 N_1 \overline{U}_1^2 \text{ and } p_2 V_2 = \tfrac{1}{3} m_2 N_2 \overline{U}_2^2.$$

If the pressures and volumes are the same,

$$m_1 N_1 \overline{U}_1^2 = m_2 N_2 \overline{U}_2^2.$$

If the temperatures are equal, then, assuming that the mean kinetic energies are the same,

$$m_1 \overline{U}_1^2 = m_2 \overline{U}_2^2,$$

so that

$$N_1 = N_2.$$

That is, *equal volumes of two gases, under the same conditions of temperature and pressure, contain the same number of molecules.*

This is the formal statement of Avogadro's hypothesis. It follows, therefore, that the weights of these equal volumes are proportional to the molecular weights of the gases, and hence that the gramme-molecular volumes of all gases, measured under the same conditions of temperature and pressure, are the same.

The gramme-molecular volume, measured at 0° C. and 760 mm. of Hg, is 22·38 litres. If then V stands for this volume, the constant R will be the same for all gases. Its value should be calculated by the reader.

But no gas behaves in this simple manner, although for moderate pressures and high temperatures the equation is accurate enough for ordinary computations, as far as the more permanent gases are concerned.

Suppose that we study experimentally the p - v relations of different fluids, drawing the isothermals for various different temperatures (fig. 14). Starting at a sufficiently low temperature we find that the volume steadily diminishes with increase of pressure up to a certain point at which the fluid separates into two phases—liquid and gaseous. The pressure then remains constant until the gaseous

* I.e. the volume occupied by M gm. of a gas at normal temperature and pressure, where M is the molecular weight.

phase has completely disappeared, when further increase of pressure causes but small diminution in volume. If we now repeat the experiment at a higher temperature, we find that the horizontal portion AB of the curve, representing the period of transition from the gaseous to the liquid phase, is shorter, and shortens steadily with increasing temperature until the isothermal for a certain temperature exhibits a point of inflexion with a horizontal tangent, running for a moment parallel to the volume-axis, and then turning upwards again. The temperature for which this isothermal is drawn is called the *critical temperature*, and the point G the critical point.* Above this temperature no amount of pressure causes a separation into two distinct phases.

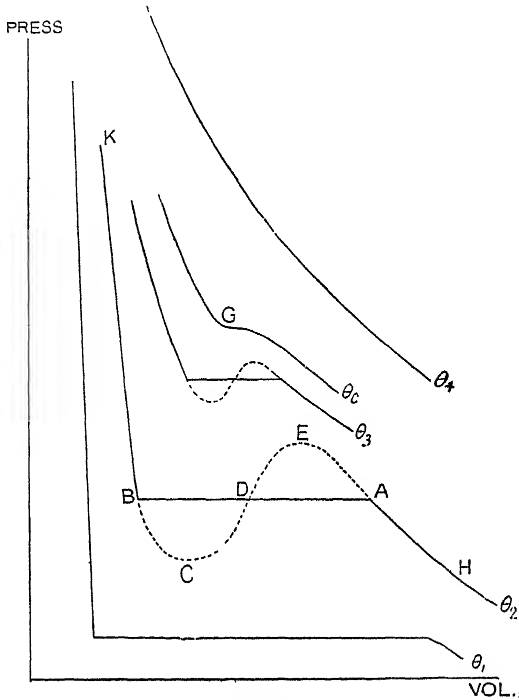


Fig. 14

The experimental determination of these curves, over a wide range of pressure and temperature, is a matter of no small difficulty. Once a suitable pressure gauge has been devised—we have seen that the change of electrical resistance of manganin may be utilized—observations are fairly straightforward, but the calibration of such a gauge demands experimental work on a heroic scale. Amagat, for example, performed a Boyle's Law experiment in which nitrogen was compressed in the closed (shorter) limb of a U-tube, the open limb being installed on the side of a shaft 327 m. deep. The p-v relation for nitrogen being known, this gas may then be used as a standard in studying the behaviour of other

* For a description of the physical state of the fluid at the critical point, consult any of the standard treatises on heat, e.g. Poynting and Thomson, or Preston.

gases, or in calibrating a different type of pressure gauge.

Let us now examine briefly the character of the curves obtained from experiments of this type. It is convenient to plot pv against p, as this procedure exhibits very clearly the departure of the gas concerned from the " perfect " state. Some of the results obtained are shown in fig. 15.

The reason for the difference between the isothermals for an ideal and for an imperfect fluid is not far to seek. The equation

$$pV = RT$$

takes no account of the forces of attraction between the molecules, nor of the volume occupied by the molecules themselves. It makes

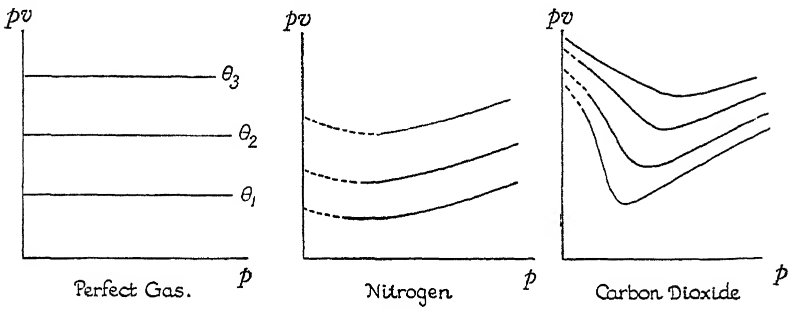


Fig. 15

V zero when p becomes indefinitely great, and it is clearly more in accordance with the properties of fluids to put

$$p(V - b) = RT,$$

so that as p increases indefinitely, V tends to the limit b; b represents, therefore, the smallest volume into which the molecules can be packed.

Further, the mutual attraction of the molecules will result in the production of a capillary pressure at the fluid surface; the intensity of the molecular bombardment will be diminished, and the pressure on the surface of the containing vessel correspondingly decreased. Put, therefore,

$$(p + \omega)(V - b) = RT.$$

Without discussing the matter very closely, we can determine the value of ω from consideration of the fact that the attraction between two elementary portions of the fluid is jointly proportional to their

masses—that is, in a homogeneous fluid, to the square of the density, or inversely as the square of the volume. We see, then, reasons for writing $\omega = \frac{a}{V^2}$, and the equation of state becomes

$$\left(p + \frac{a}{V^2}\right)\left(V - b\right) = RT,$$

the form originally proposed by van der Waals.

This equation is a cubic in v, and if the isothermals are plotted for different values of θ, we obtain curves whose general shape is that of the curve HAEDCBK of fig. 14. It will be observed that for temperatures below the critical temperature a horizontal constant pressure line cuts any given isothermal either in one point or in three points—corresponding to the roots of the van der Waals cubic. Taking an isothermal nearer to the critical temperature, we see that the three real roots are more nearly coincident, and at G, the critical point itself, the roots coincide. Above the critical point, a horizontal line cuts any given isothermal in one point only—two of the roots of the cubic are imaginary. If we write down the condition that the three roots shall be coincident, we easily arrive at values of the critical constants in terms of the constants of van der Waals' equation. These are

$$v_c = 3b, \quad p_c = \frac{a}{27b^2}, \quad T_c = \frac{8a}{27Rb}.$$

But it is preferable to write down the condition that at the critical point the isothermal has a point of inflexion with a horizontal tangent. If we therefore differentiate van der Waals' equation with respect to v, put $\frac{\partial p}{\partial v}$ and $\frac{\partial^2 p}{\partial v^2}$ equal to zero, the resulting equations, combined with the original equation of state, serve to determine p_c, v_c, and T_c. The work is left as an exercise to the reader.

This method is preferable, since it is perfectly general and may be applied to characteristic equations which are not cubics in v, and to which, therefore, the "equal-root" method, beloved of writers on physical chemistry, is not applicable.

It will be observed that the equation tells us nothing concerning the straight line AB, which represents the actual passage observed in nature from the vapour to the liquid phase. The position of this line on any given isothermal can, however, be obtained from the simple consideration that the areas DBCD and AEDA must be

equal,* and the line must be drawn to fulfil this condition.

Fig. 15 shows that the pv-p curves in general exhibit a minimum value for pv, and that the locus of these points lies on a definite curve. The equation to this curve may be obtained by writing pv as y and p as x in the characteristic equation, and expressing the condition that y should have a minimum value.

Of the other characteristic equations that have from time to time been proposed we may cite, naming them by their authors:

Clausius (a):

$$\left(p + \frac{a}{\mathrm{TV}^2}\right)(\mathrm{V} - b) = \mathrm{RT};$$

Clausius (b):

$$\left(p + \frac{a}{\mathrm{T}(\mathrm{V}+\mathrm{C})^2}\right)(\mathrm{V} - b) = \mathrm{RT};$$

Dieterici (a):

$$\left(p + \frac{a}{\mathrm{V}^k}\right)(\mathrm{V} - b) = \mathrm{RT},$$

Dieterici (b):

$$p(\mathrm{V} - b) = \mathrm{RT}e^{-\frac{\mathrm{A}}{\mathrm{RTV}}}.$$

The deduction of the critical constants from these equations is left to the reader.

The value of a characteristic equation which shall closely represent the pressure-volume relations of a fluid over a wide range of pressures and temperatures, is obvious. We have seen, in the section on compressibility, that many important physical constants may be expressed in terms of thermodynamic equations involving certain differential coefficients and integrals. The values of these physical constants may be worked out by substituting, in the appropriate thermodynamic equations, the values of the differential coefficients

* For in any reversible cycle $(\int)\frac{dQ}{T} = 0$. If the cycle be isothermal,

$$(\int)\frac{dQ}{T} = \frac{1}{T}(\int)\,dQ, \text{ and therefore } (\int)dQ = 0.$$

But for *any* cycle

$$(\int)(dQ + dW) = 0,$$

and hence for an isothermal reversible cycle $(\int)dW = 0$. So that, if we take unit mass of the substance reversibly round the cycle AEDCBDA (fig. 14), the work done, represented by the sum of the positive and negative areas AEDA and DCBD, must be zero. Hence the two areas are equal. (For critical remarks on this proof see Preston, *Theory of Heat*, 479 (1904), or Jeans, *Dynamical Theory of Gases*, p. 159.)

obtained from the differentiation of the equation of state. Unfortunately, no equation yet proposed covers the whole ground satisfactorily. An equation which fits the experimental figures at one end of the scale is usually unsatisfactory at the other end, and conversely.

A few simple tests may be suggested by which the fitness of any given equation may be roughly examined. The experiments of Professor Young show that, while the value of the ratio $\frac{RT_c}{p_c v_c}$ varies slightly from substance to substance, its mean value may be taken as about 3·75. Now the equation of state of an ideal gas gives unity for this ratio, and is clearly very far out of it. Van der Waals' equation gives

$$\frac{RT_c}{p_c v_c} = \left(R \times \frac{8a}{27Rb}\right) \div \left(\frac{a}{27b^2} \times 3b\right) = 2{\cdot}67,$$

and is not a very good approximation to the truth.

Similarly the (*a*) and (*b*) equations of Clausius give for this ratio the values 2·67 and 3·00 respectively, and the corresponding equations of Dieterici give 3·75 and 3·69, if in the (*a*) equation we take the value of k as $\frac{5}{3}$.

Another test may be derived from the experimental fact that the critical specific volume v_c is about four times the liquid volume. Now the constant b, which represents the least volume into which the molecules can be packed, cannot be seriously different from the liquid volume. Accordingly we find, on working out the values for v_c, that for the (*a*) and (*b*) equations of Clausius the values of v_c are $3b$ and $4b$ respectively,* while the corresponding equations of Dieterici give the values $4b$ and $2b$. For van der Waals' equation the value is, as we have seen, $3b$.

But the subject may be studied from a different point of view. Instead of attempting to devise an equation which shall represent the properties of a substance over a wide range—a process which usually results in a cumbrous formula—we may try to arrive at an equation which shall be simple and manageable in form, so that the various physical constants of the fluid may be readily worked out from the corresponding thermodynamic relations, while at the same time the equation shall represent a very close approximation to the truth over a *limited* range, the range chosen being one of practical importance. Whether such a formula can, or cannot, be extrapolated

* If in the (*b*) equation we put $2C = b$.

beyond the limits of the range is a matter of secondary interest—what is important is that the formula should be as exact as may be within these limits.

Let us then investigate the form which such an equation as the (b) equation of Clausius assumes for moderate pressures. Re-writing the equation as

$$p(V - b) = RT - \frac{a(V - b)}{T(V + c)^2},$$

we see that at moderate pressures, when the volume is large, we shall not be seriously in error if we write

$$\frac{V - b}{(V + c)^2} = \frac{1}{V} \text{ (approximately).}$$

We thus have

$$p(V - b) = RT - \frac{a}{TV};$$

and again, putting, in the small term,

$$V = \frac{RT}{p},$$

we find, on rearranging the equation,

$$V = \frac{RT}{p} - \frac{c}{T^2} + b,$$

where c is put for $\frac{a}{R}$.

If we replace T^2 by the more general form T^n, where n varies from substance to substance, we have

$$V = \frac{RT}{p} - \frac{c}{T^n} + b,$$

which is the form known as Callendar's equation.

This equation has been applied very successfully to elucidate the properties of steam over a range of pressure from 0 to 34 atmospheres; the value of n appropriate to steam is $\frac{10}{3}$. Space will not permit us to discuss at length this important equation. Indeed, the discussion lies within the province of thermodynamics, and the reader desirous of further information should consult the articles "Thermodynamics" and "Vaporization" in the *Encyclopædia Britannica*, or a textbook such as Ewing's *Thermodynamics for Engineers*.

Osmotic Pressure

If we throw a handful of currants or raisins into water and leave them for a while, we find that the fruits, originally shrunken and wrinkled, have swelled out and become smooth.* Water has passed through the skin of the fruit, while the dissolved substances inside cannot pass out—or at least do not stream out so freely as the water streams in. This unilateral passage of a substance through a membrane is termed *osmosis*.†

In the limiting case, when we have a solution on one side of a membrane and pure solvent on the other, the membrane is called a semi-permeable membrane if it freely admits of the passage of the solvent, but is strictly impervious to the dissolved substance. It has been asserted that no such membranes exist in nature, but, as far as experiment can show, a membrane of copper ferrocyanide forms a true semi-permeable membrane to a solution of sugar in water.

Suppose such a membrane, prepared with due precautions ‡—and it is not so easy as one would imagine to prepare a thoroughly resistant membrane—to be deposited on the inside of a cylindrical porous pot. The pot is filled with a sugar solution, closed, and attached to a suitable manometer which shall measure the pressure inside the pot. It is then placed in a vessel containing pure water. We shall find that the pressure in the pot rises, finally reaching a maximum stable value. The maximum value of this pressure, *assuming that the membrane is truly semi-permeable*, is called the osmotic pressure of the solution.

It will thus be seen that osmotic pressure is defined in terms of a semi-permeable membrane. It is only in very loose phraseology that one can speak of the osmotic pressure of a solution without reference to the existence of a semi-permeable membrane. A solution, *qua* solution, has no osmotic pressure.

If this definition be consistently followed, a great deal of vague and loose reasoning of the soporific-power-of-opium variety will be swept away. It is all the more needful to emphasize this point as there has arisen, in biological (and even in engineering) circles, a tendency to ascribe to " osmotic pressure " a power and potency

* Imbibition of water by the dried tissues will also play a part in the smoothing process.

† From ὠσμός, a rare Greek noun meaning " thrusting " or " pushing through ".

‡ Morse, *Jour. Amer. Chem. Soc.*, **45**, 91 (1911).

which is almost proportional to the vagueness with which the mechanism of that pressure is conceived.

Consider, for example, the common remark that osmotic pressure "acts the wrong way"—that is, causes motion from a region of lower osmotic pressure to a region of higher osmotic pressure. It only requires a little consideration of the definition of osmotic pressure fully to realize that the argument involves a ὕστερον πρότερον, for it is clear that it is *osmosis* which produces osmotic pressure, not osmotic pressure which produces osmosis.

The quantitative laws of osmotic pressure were first studied by Pfeffer, whose figures show that for dilute solutions the pressure, at constant temperature, is proportional to the concentration, and that at constant concentration the pressure is proportional to the absolute temperature. We may therefore write

$$\mathrm{PV} = \mathrm{KT},$$

and it has been shown by van't Hoff that the constant K has the same value as the gas constant R. Hence it follows that the osmotic pressure of the solution is the same as that which would be exerted by the dissolved substance were it dispersed in the form of a gas through a volume equal to that occupied by the solution.

If we desire to correct this simple gas law, we find it necessary to look at the matter from a different angle. We *define* an ideal solution as containing two completely miscible unassociated components, of such a nature that there occurs no change of volume on mixing, and that the heat of dilution is negligible.

For such a solution it can be shown that the osmotic pressure P is given by

$$\mathrm{P} + \beta\frac{\mathrm{P}^2}{2} = \frac{\mathrm{RT}}{\mathrm{V}}\left\{-\log_e(1 - x)\right\},$$

where β denotes the compressibility of the solvent, V its molecular volume, and x the ratio of the number of molecules of the dissolved substance to the total number present. If we neglect β, which is usually small, and expand the logarithmic term, we have the convenient form

$$\mathrm{P} = \frac{\mathrm{RT}}{\mathrm{V}}\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots\right).$$

This equation holds good for any concentration.

Despite a large amount of criticism, the kinetic theory of osmotic pressure still holds the field as the only one which gives values of the

pressure calculated from theory.* The properties of the membrane, which play a large part in some theories, whilst of great interest and value, are distinctly of secondary importance in the kinetic theory. It is the thermal agitation of the molecules of the solute which is effective in producing osmotic pressure, and the magnitude of the pressure calculated from the agitation of the molecules is equal to the value obtained by experiment. "Any other theory put forward to account for osmosis must fulfil, then, a double duty; not only must it be competent to explain osmosis, but it must also explain *away* the effects that we have the right to expect from the molecular agitation of the solute."†

NOTE ON TRANSFORMATION OF CO-ORDINATES

Perhaps the simplest way of passing from one form to the other is to consider the concentration of fluid, reckoned per unit volume per second, at a point.

If u, v, w are the component velocities at a point P, the fluid leaving the elemental volume $\delta x \delta y \delta z$ in time δt is readily found, by the method given below for a more difficult case, to exceed that which enters it, by an amount

$$\rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\delta x \delta y \delta z \delta t,$$

where ρ is the density, i.e. the concentration, reckoned in mass per unit volume per second, is

$$-\rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right).$$

Taking now the element of volume shown in fig. 16,

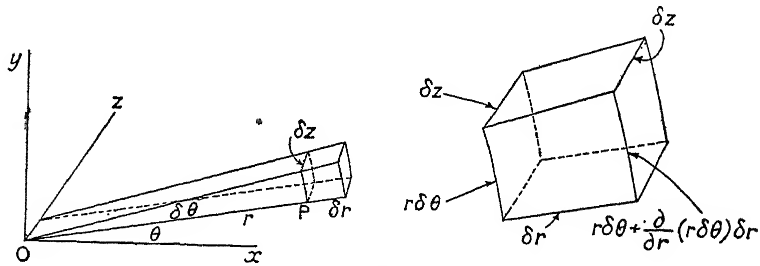


Fig. 16

* Porter, *Trans. Far. Soc.*, **13**, 10 (1917). The reader desiring more information on the subject should consult this valuable discussion. † Porter, loc. cit., p. 8.

and letting u' be the radial velocity and v' the tangential velocity at P, we get

$$u'r\delta\theta\delta z\delta t \text{ for the radial flow in, and}$$

$$\left(u' + \frac{\partial u'}{\partial r}\delta r\right)\left(r\delta\theta + \frac{\partial}{\partial r}(r\delta\theta)\delta r\right)\delta z\delta t \text{ for the radial flow out.}$$

The latter exceeds the former by

$$\left(r\frac{\partial u'}{\partial r} + u'\right)\delta\theta\delta r\delta z\delta t.$$

The tangential flow in is $v'\delta r\delta z\delta t$,
and the tangential flow out,

$$\left(v' + \frac{\partial v'}{\partial \theta}\delta\theta\right)\delta r\delta\theta\delta z\delta t,$$

which exceeds the former expression by

$$\frac{\partial v'}{\partial \theta}\delta\theta\delta r\delta z\delta t,$$

i.e the total flow out is

$$\left(u' + r\frac{\partial u'}{\partial r} + \frac{\partial v'}{\partial \theta}\right)\delta\theta\delta r\delta z\delta t,$$

i.e. the total flow *into* the element is this expression taken with the minus sign.

The elemental volume is $r\delta\theta\delta z\delta r$;
hence the concentration is

$$-\rho\left(\frac{u'}{r} + \frac{\partial u'}{\partial r} + \frac{1}{r}\frac{\partial u'}{\partial \theta}\right).$$

The concentration—mass per unit volume per second—must be the same whatever co-ordinates we use, hence

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \equiv \frac{u'}{r} + \frac{\partial u'}{\partial r} + \frac{1}{r}\frac{\partial u'}{\partial \theta} \qquad \text{(1)}$$

If the fluid motion has a velocity potential, the component velocity in any direction is the gradient of potential in that direction, i.e.

$$u = \frac{\partial \phi}{\partial x}, \quad v = \frac{\partial \phi}{\partial y},$$

$$u' = \frac{\partial \phi}{\partial r}, \quad v' = \frac{1}{r}\frac{\partial \phi}{\partial \theta},$$

for the element of length perpendicular to r, i.e. in the tangential direction, is $rd\theta$; hence

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\equiv\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\cdot\frac{\partial^2\phi}{\partial\theta^2},\qquad\text{(2)}$$

by substituting the values of u, u', v, v', in equation (1).

We have *proved* the transformation when the dependent variable is the velocity potential ϕ, but as the transformation is a purely analytical one in its nature, the form must be equally true whatever be the physical nature of ϕ; for instance, if $\phi\equiv w$, the z component of velocity,

$$\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\equiv\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2},$$

the form which actually occurs (p. 33), and, in general,

$$\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}\equiv\frac{\partial^2 U}{\partial r^2}+\frac{1}{r}\frac{\partial U}{\partial r}+\frac{1}{r^2}\frac{\partial^2 U}{\partial\theta^2},\qquad\text{(3)}$$

at every point in a plane where U is a single-valued function of the co-ordinates of the point and possesses finite derivatives up to those of the second order there.

CHAPTER II

Mathematical Theory of Fluid Motion

It is assumed throughout this chapter that the fluid with which we deal may be regarded as incompressible. This only means that changes of pressure are propagated in it instantaneously, instead of with the (very great) velocity of sound. Since the velocity of sound in air is only about four times less than in water, it is clear that many of our results will be equally applicable to gaseous fluids, the influence of compressibility being negligible except in the case of very rapid differential motions.

It is further assumed in the first instance that the fluid is *frictionless*, i.e. that it presses perpendicularly on any surface with which it is in contact, whether it be the surface of an adjacent portion of fluid, or of a solid boundary. This hypothesis of the absence of all tangential stress is not in accordance with fact, but it greatly simplifies the mathematics of the subject, and there are, moreover, many cases of motion in which the influence of friction is only secondary. It follows from this assumption that the state of stress at any point P of the fluid may be specified by a single quantity p, called the "pressure-intensity" or simply the "pressure", which measures the force per unit area exerted on any surface through P, whatever its aspect. This is in fact the cardinal proposition of hydrostatics.

It is convenient here to prove, once for all, that the resultant of the pressures exerted on the boundary of any small volume Q of fluid is a force whose component in the direction of any line element δs is

$$-\frac{\partial p}{\partial s}\mathrm{Q}, \qquad \text{(1)}$$

where $\partial p/\partial s$ is the gradient of p in the direction of δs. Take first the case of a columnar portion of fluid whose length δx is parallel to

the axis of x, and suppose that the dimensions of the cross-section (ω) are small compared with δx. The pressure-intensities at the two ends may then be denoted by

$$p \text{ and } p + \frac{\partial p}{\partial x}\delta x,$$

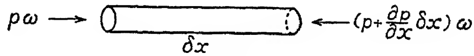


Fig. 1

so that the component parallel to the length of the pressure on the column is

$$p\omega - \left(p + \frac{\partial p}{\partial x}\delta x\right)\omega = -\frac{\partial p}{\partial x}\omega\delta x,$$

the pressures on the sides being at right angles to δx. Since $\omega\delta x$ is the volume of the column, the formula (1) is in this case verified. Since, moreover, *any* small volume Q may be conceived as built up of columnar portions of the above kind, and since there is nothing special to the direction Ox, the result is seen to be general.

Stream-line Motion

1. *Bernoulli's Equation*

A state of *steady* or *stream-line* motion is one in which the stream-lines, i.e. the actual paths of the particles, preserve their configuration unchanged. The most obvious examples are where a stream flows past a stationary solid; and the designation is naturally extended to cases where a solid moves uniformly in a straight line, without rotation, through a surrounding fluid, provided the superposition of a uniform velocity equal and opposite to that of the solid reduces the case to one of steady motion in the former sense. This super-posed velocity does not of course make any difference to the dynamics of the question.

The stream-lines drawn through the contour of any small area will mark out a tube, which we may call a *stream-tube*. Since the same volume of fluid must traverse each cross-section in the same time, we have

$$q_1\omega_1 = q_2\omega_2,$$

where ω_1, ω_2 are the areas of any two cross-sections at P_1, P_2, and q_1, q_2 the corresponding velocities in the direction (say) from P_1 to P_2. Now consider the region included between these two sections. In a short time dt a volume Q $= \omega_1 q_1 dt$ will have entered it at

P_1, and an equal volume $Q = \omega_2 q_2 dt$ will have left it at P_2. The work done by hydrostatic pressure in this time on the mass of fluid which originally occupied the space P_1P_2 will be $p_1\omega_1 q_1 dt$ or p_1Q_1, at P_1, and $-p_2\omega_2 q_2 dt$ or $-p_2Q$ at P_2. The same mass will have gained kinetic energy of amount $\frac{1}{2}\rho Q(q_2{}^2-q_1{}^2)$, where ρ is the density, i.e. the mass per unit volume. If V denotes the potential energy of unit mass, the gain of potential energy will be $\rho Q(V_2 - V_1)$. Hence, equating the work done on the mass to the total increment of energy we have

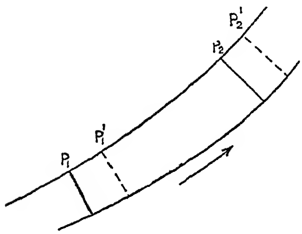


Fig. 2

$$p_1 - p_2 = \tfrac{1}{2}\rho(q_2{}^2 - q_1{}^2) + \rho(V_2 - V_1),$$

or
$$p_1 + \tfrac{1}{2}\rho q_1{}^2 + \rho V_1 = p_2 + \tfrac{1}{2}\rho q_2{}^2 + \rho V_2.$$

Hence along any stream-line

$$p + \tfrac{1}{2}\rho q^2 + \rho V = C, \quad\ldots\ldots\ldots\ldots\ldots \quad (2)$$

where C is a constant for that particular line, but may vary from one stream-line to another. This equation is due to D. Bernoulli (1738) and was proved by him substantially in the above manner.

The formula has many applications. For instance, in the case of water issuing from a small orifice in the wall of an open vessel we have at the upper surface $p = p_0$ (the atmospheric pressure), and $q = 0$, approximately. Again, the value of V at the upper surface exceeds that at the orifice by gz, where z is the difference of level and g the acceleration due to gravity. Hence if q be the velocity at the surface of the issuing jet,

$$p_0 + g\rho z = p_0 + \tfrac{1}{2}\rho q^2,$$

or
$$q^2 = 2gz, \quad\ldots\ldots\ldots\ldots\ldots (3)$$

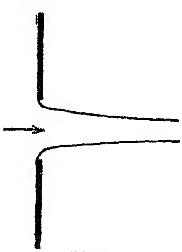


Fig. 3

a formula due to Torricelli (1643). If S′ be the section of the jet at the "vena contracta", where it is sensibly parallel, the pressure over S′ will be p_0. The velocity will therefore be given by the above value of q, and the discharge per unit time will be $\rho q S'$. The ratio of S′ to the area S of the orifice is called the "coefficient of contraction". It is not easy to determine this coefficient theoretically, but a

very simple argument shows that in the case of an orifice in a thin wall it must exceed $\frac{1}{2}$. Take, for instance, an orifice in a vertical wall. In every second a mass of $\rho q S'$ escapes with the velocity q, and carries with it a momentum $\rho q^2 S'$. This represents the horizontal force exerted by the vessel on the fluid. There must be a contrary reaction of this amount on the vessel. On the opposite wall of the vessel, where the velocity is insignificant, the pressure has sensibly the statical value due to the depth, and if this were also the case on the wall containing the orifice there would be an unbalanced force $g\rho z S$ urging the vessel backwards. Actually, owing to the appreciable velocity, the pressure near the orifice will be somewhat less, so that the reaction exceeds $g\rho z S$. Hence $\rho q^2 S' > g\rho z S$, or (since $q^2 = 2gz$) $S' > \frac{1}{2}S$. In a particular case, where the fluid escapes by a tube projecting inwards in the manner shown in the figure, the statical pressure obtains practically over the walls, and $S' = \frac{1}{2}S$, exactly. This arrangement is known as "Borda's mouthpiece".

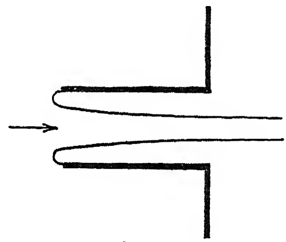


Fig. 4

Another application of (3), much used in aeronautics and engineering, is to the measurement of the velocity of a stream, e.g. of the relative wind in an aeroplane. The quantities p and $p + \frac{1}{2}\rho q^2$ are measured independently, and their difference determines q. A fine tube, closed at one end and connected with a pressure-gauge at the other, points up the stream so as to interfere as little as possible with the motion, and contains a few minute holes in its side, at a little distance from the closed end; the gauge therefore gives the value of p. On the other hand, an open tube drawn out at the end almost to a point, and connected to a second gauge, will give the value of $p + \frac{1}{2}\rho q^2$ at a short distance ahead of the vertex. For if p' be the pressure at the vertex itself, where the velocity is arrested, we have $p + \frac{1}{2}\rho q^2 = p'$, for points on the same stream-line. The two contrivances are often united (as in the figure) in a single appliance known as a "Pitot and static pressure tube".

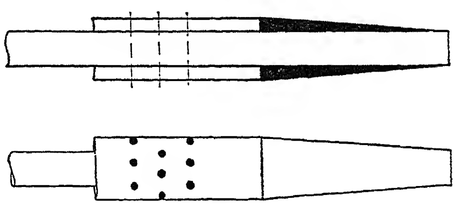


Fig. 5

2. *Two-dimensional Motion. Stream-function*

There are two types of stream-line motion which are specially simple and important. We take first the two-dimensional type, where the motion is in a system of parallel planes, and the velocity has the same magnitude and direction at all points of any common normal. It is sufficient then to confine our attention to what takes place in one of these planes. Any line drawn in it may be taken to represent the portion of the cylindrical surface, of which it is a cross-section, included between this plane and a parallel plane at unit distance from it. By the " flux " across the line we understand the volume of fluid which in unit time crosses the surface thus defined. Now taking an arbitrarily fixed point A and a variable point P, the flux (say from right to left) will be the same across any two lines drawn from A to P, provided the space between them is wholly occupied by fluid. This is in virtue of the assumed constancy of volume. The flux will therefore be a function only of the position of P; it is usually denoted by the letter ψ. It is evident at once from the definition that the value of ψ will not alter as the point P describes a stream-line, and therefore that the equation

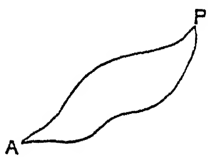


Fig. 6

$$\psi = \text{constant} \quad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

will define a stream-line. For this reason ψ is called the *stream-function*. If P′ be any point adjacent to P, the flux across AP′ will differ from that across AP by the flux across PP′; whence, writing PP′ $= \delta s$, we have $\delta\psi = q_n \delta s$, where q_n is the component velocity normal to PP′, to the left. Thus

$$q_n = \frac{\partial\psi}{\partial s}, \quad \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

where $\partial\psi/\partial s$ is the gradient of ψ in the direction of δs. This leads to expressions for the component velocities u, v parallel to rectangular co-ordinate axes. If we take δs parallel to Oy we have $q_n = -u$, whilst if it be taken parallel to Ox we have $q_n = v$. Thus

$$u = -\frac{\partial\psi}{\partial y}, \quad v = \frac{\partial\psi}{\partial x}. \quad \ldots\ldots\ldots\ldots\ldots (6)$$

These satisfy the relation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots (7)$$

which is called the *equation of continuity*. It may be derived otherwise by expressing that the total flux across the boundary of an elementary area $\delta x \delta y$ is zero. Again, if we use polar co-ordinates r, θ, and take δs ($= \delta r$) along the radius vector, we have $q_n = v$, where v denotes the transverse velocity; whilst if δs ($= r\delta\theta$) be at right angles to r, $q_n = -u$, the radial velocity. Hence

$$u = -\frac{\partial\psi}{r\partial\theta}, \quad v = \frac{\partial\psi}{\partial r}. \qquad \text{(8)}$$

It follows that

$$\frac{\partial}{\partial r}(ru) + \frac{\partial v}{\partial\theta} = 0, \qquad \text{(9)}$$

which is another form of the equation of continuity.

It is to be remarked that the above definition of ψ is purely geometrical, and is merely a consequence of the assumed incompressibility of the fluid. If we make any assumption whatever as to the form of this function, the formulæ (6) or (8) will give us a possible type of motion; but it by no means follows that it will be a possible type of permanent or steady motion. To ascertain the condition which must be fulfilled in order that this may be the case we must have recourse to dynamics; but before doing this it is convenient to introduce the notions of *circulation* and *vorticity*.

The *circulation* round any closed line, or circuit, in the fluid is the line-integral of the tangential velocity taken round the curve in a prescribed sense. In symbols it is

$$\int q \cos\chi \, ds, \qquad \text{(10)}$$

where χ is the angle which the direction of the velocity q makes with that of the line-element δs. In rectangular co-ordinates, resolving u and v in the direction of δs, we have

$$q \cos\chi = u\frac{dx}{ds} + v\frac{dy}{ds},$$

so that the circulation is

$$\int\left(u\frac{dx}{ds} + v\frac{dy}{ds}\right)ds, \quad \text{or} \quad \int(udx + vdy). \qquad \text{(11)}$$

It will appear that the circulation round the contour of an infinitely small area is ultimately proportional to the area. The ratio which it bears to the area measures (in the present two-dimensional case) the *vorticity*; we denote it by ζ. Its value, in terms of rectangular

co-ordinates, is found by calculating the circulation round an elementary rectangle PQRS whose sides are δx and δy. The portions of the line-integral (11) due to PQ and RS are together equal to the difference in the corresponding values of $u\delta x$, i.e. to $-\frac{\partial u}{\partial y}\delta y\delta x$. The portions due to QR and SP are in like manner equal to $\frac{\partial v}{\partial x}\delta x\delta y$. Equating the sum to $\zeta\delta x\delta y$, we have

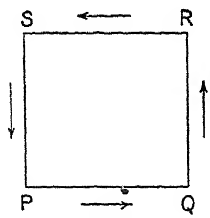


Fig. 7

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}, \qquad\text{.................(12)}$$

or, from (6),

$$\zeta = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}. \qquad\text{...............(13)}$$

The value of ζ at any point P is related to the average rotation relative to P of the particles in the immediate neighbourhood. To examine this, we calculate the circulation round a circle of small radius r having P as centre. The velocity at any point Q on the circumference may be regarded as made up of a general velocity equal to that of P, and the velocity relative to P. The former of these contributes nothing to the required circulation. The latter gives a tangential component ωr, where ω is the angular velocity of QP. The circulation is therefore

$$\int_0^{2\pi} \omega r . r d\theta = r^2\int_0^{2\pi} \omega d\theta,$$

where θ is the angular co-ordinate of Q. Since the same thing is expressed by $\zeta\pi r^2$, we have

$$\zeta = \frac{1}{\pi}\int_0^{2\pi} \omega d\theta, \qquad\text{.................(14)}$$

which is *twice* the average value of ω on the circumference. For this reason a type of motion in which ζ is everywhere zero, i.e. in which the ratio of the circulation in every *infinitesimal* circuit to the area within the circuit vanishes, is called *irrotational.*

3. *Condition for Steady Motion*

We can now ascertain the dynamical conditions which must be satisfied in order that a given state of motion may be steady. For this purpose we consider the forces acting on an element, PQQ′P′, of fluid included between two adjacent stream-lines and two adjacent normals. The latter meet at the centre of curvature C. We write PQ $= \delta s$, PP′ $= \delta n$, PC $=$ R. The mass of the element is therefore $\rho\delta s\delta n$. The forces acting on it may be resolved in the direction of the tangent and normal, respectively, to the stream-line PQ. Tangential resolution would merely lead us again, after integration, to Bernoulli's equation (2). Normal resolution gives, with the help of (1),

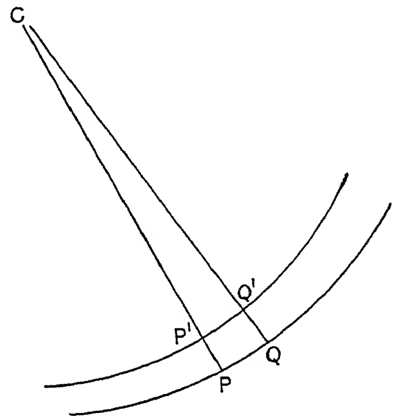


Fig. 8

$$\rho\delta s\delta n\frac{q^2}{R} = -\frac{\partial p}{\partial n}\delta s\delta n - \frac{\partial V}{\partial n}\rho\delta s\delta n,$$

where $\partial p/\partial n$ and $\partial V/\partial n$ are the gradients of p and V in the direction PC. Hence

$$\frac{\partial}{\partial n}(p + \rho V) = -\frac{\rho q^2}{R}. \qquad (15)$$

The circulation round the circuit PQQ′P′ will be equal to $\zeta\delta s\delta n$. In calculating this circulation, we may neglect the sides PP′, QQ′ since they are at right angles to the velocity. The contributions of the remaining sides are

$$q\delta s \quad \text{and} \quad -\left(q + \frac{\partial q}{\partial n}\delta n\right)\delta s',$$

where $\delta s' =$ P′Q′. Now from the figure we have

$$\frac{\delta s'}{\delta s} = \frac{CP'}{CP} = \frac{R - \delta n}{R},$$

so that the circulation is

$$q\delta s - \left(q + \frac{\partial q}{\partial n}\delta n\right)\left(1 - \frac{\delta n}{R}\right)\delta s = \left(\frac{q}{R} - \frac{\partial q}{\partial n}\right)\delta s\delta n,$$

omitting terms of higher order than those retained. Hence

$$\zeta = -\frac{\partial q}{\partial n} + \frac{q}{R}. \quad \ldots\ldots\ldots\ldots\ldots\ldots (16)$$

The formula (15) may now be written

$$\frac{\partial}{\partial n}(p + \tfrac{1}{2}\rho q^2 + \rho V) = -\rho q \zeta. \quad \ldots\ldots\ldots\ldots (17)$$

Comparing this with (2), we have

$$\delta C = -\rho q \zeta \delta n, \quad \ldots\ldots\ldots\ldots\ldots\ldots (18)$$

where C is the quantity which was proved before to be constant along any stream-line, but will in general vary from one stream-line to another. If we fix our attention on two consecutive stream-lines, δC will be a constant, and $q\delta n$ will also obviously be constant. The dynamical condition for steady motion is therefore that the vorticity should be constant along any stream-line. When it is fulfilled, the distribution of pressure is given by (2) and (17). We may express the result otherwise by saying that any fluid element retains its vorticity unchanged as it moves along. This is a particular case of a theorem in vortex-motion to be proved later.

An obvious example is that of fluid rotating with uniform angular velocity ω about a vertical axis, and subject to gravity. The law of distribution of pressure may be deduced from (17), or more simply from first principles. If r be the radius of the circular path of a small volume Q, the resultant force upon it must be radial, of amount $\rho Q \omega^2 r$. Hence, and since there is no vertical acceleration, we have

$$\rho\omega^2 r = \frac{\partial p}{\partial r}, \quad 0 = -\frac{\partial p}{\partial z} - g\rho, \quad \ldots\ldots\ldots (19)$$

the positive direction of z being that of the upward vertical. It follows that

$$p = \tfrac{1}{2}\rho\omega^2 r^2 - g\rho z + \text{constant}. \quad \ldots\ldots\ldots (20)$$

The free surface ($p = p_0$) is therefore the parabola

$$z = \frac{\omega^2}{2g} r^2. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (21)$$

if the origin of z is where the free surface meets the axis ($r = 0$).

If we imagine the fluid contained within the cylindrical surface $r = a$, rotating in the above manner, to be surrounded by fluid moving irrotationally, we have in the latter region $\partial q/\partial r + q/r = 0$, from (16), or

$$qr = \text{constant} = \omega a^2. \qquad (22)$$

Bernoulli's equation then gives

$$p = \text{constant} - g\rho z - \tfrac{1}{2}\rho\frac{\omega^2 a^4}{r^2}. \qquad (23)$$

The equation to the free surface is therefore

$$z = -\frac{\omega^2 a^4}{2gr^2} + \frac{\omega^2 a^2}{g}, \qquad (24)$$

where the additive constant has been chosen so as to agree with (21) when $r = a$. It appears that these equations also give the

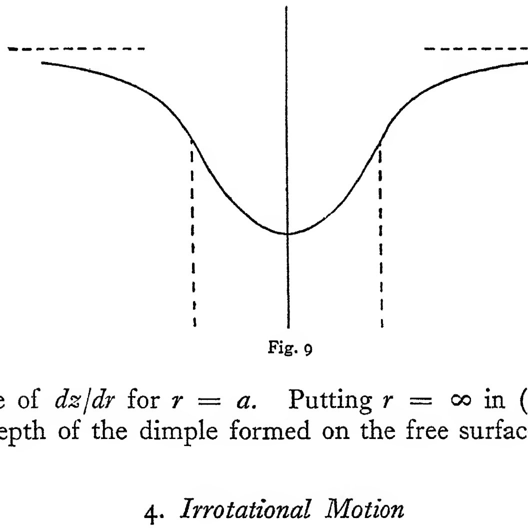


Fig. 9

same value of dz/dr for $r = a$. Putting $r = \infty$ in (24), we find that the depth of the dimple formed on the free surface is $\omega^2 a^2/g$.

4. *Irrotational Motion*

We proceed now to consider more particularly the case of irrotational motion. The condition for steady motion is fulfilled automatically if $\zeta = 0$ everywhere, provided, of course, the necessary boundary conditions are satisfied, as they are in the case of the flow of a liquid past a stationary solid. The geometrical condition (13) reduces to

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0, \qquad (25)$$

or, in polar co-ordinates,

$$\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\delta r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} = 0 \qquad (26)$$

and the pressure-distribution is such that

$$p + \tfrac{1}{2}\rho q^2 + \rho V = \text{constant} \qquad (27)$$

throughout the fluid. The particular value of the constant is for most purposes unimportant, since the addition of a uniform pressure throughout does not alter the resultant force on any small element of fluid, or on an immersed solid.

Some simple solutions of (25) or (26) are easily obtained. Thus

$$\psi = -Uy = -Ur\sin\theta \qquad (28)$$

gives a uniform flow with velocity U from left to right. Again, take the case of symmetrical radial flow outwards from the origin. The stream-lines are evidently the radii, so that ψ is a function of θ only. Since the total flux outwards across any circle r = constant must be the same, we have from (8)

$$-\frac{\partial\psi}{r\partial\theta}.2\pi r = \text{constant} = m, \text{ say,}$$

or

$$\psi = -\frac{m}{2\pi}\theta. \qquad (29)$$

If m be positive we have here the fictitious conception of a *line-source* which emits fluid at a given rate. If m be negative we have a *sink*. Since (29) would make the velocity infinite at the origin, these imaginary sources and sinks must be external to the region occupied by the fluid. The formula (29), for instance, would be realized by the expansion of a circular cylinder whose axis passes through O. Again, since the differential equations (25) and (26) are linear, they are satisfied by the sum of any number of separate solutions. For instance, the combination of a source at A, and an equal sink at a point B to the left of A, gives

$$\psi = -\frac{m}{2\pi}(\theta_1 - \theta_2), \qquad (30)$$

where θ_1, θ_2 are the angles which the lines drawn from A and B to any point P make with the direction BA. Since $\theta_1 - \theta_2 = $ APB, the lines ψ = constant are a system of circles through A, B. This

kind of motion would involve infinite velocities at A and B, but if we combine (28) with (30) we get the flow past an oval cylinder which encloses the imaginary source and sink. If the points A and B be made to approach one another, whilst m increases so that the product mAB is constant, we have ultimately $\theta_1 - \theta_2 = \text{APB} = \text{AB}\sin\theta/r$. We thus get the form

$$\psi = \mathrm{C}\frac{\sin\theta}{r} \quad \ldots\ldots\ldots\ldots\ldots\ldots (31)$$

Combining this with (28), we have

$$\psi = \left(-\mathrm{U}r + \frac{\mathrm{C}}{r}\right)\sin\theta. \quad \ldots\ldots\ldots\ldots (32)$$

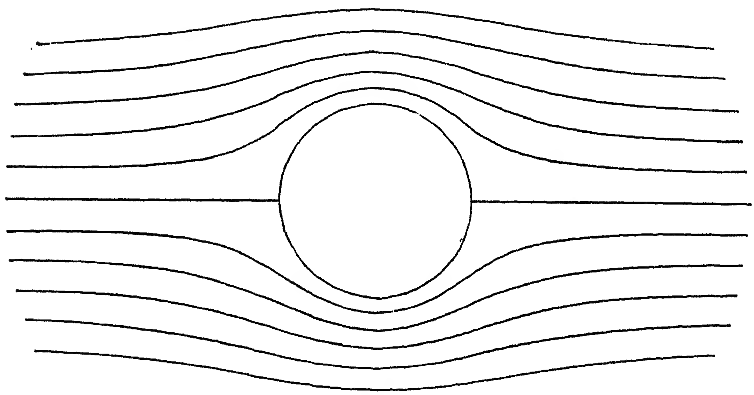


Fig. 10

The stream-line $\psi = 0$ now consists of the radii $\theta = 0$, $\theta = \pi$, and the circle $r = a$, provided $\mathrm{C} = \mathrm{U}a^2$. The formula

$$\psi = -\mathrm{U}\left(r - \frac{a^2}{r}\right)\sin\theta \quad \ldots\ldots\ldots\ldots (33)$$

therefore gives the flow past a circular cylinder. The normal velocity at the surface is of course zero, whilst the tangential velocity is

$$\frac{\partial\psi}{\partial r} = -\mathrm{U}\left(1 + \frac{a^2}{r^2}\right)\sin\theta = -2\mathrm{U}\sin\theta. \quad \ldots\ldots (34)$$

Omitting the external forces, if any, represented by V, which have merely an effect analogous to buoyancy, the pressure at the cylinder is

$$p = \text{constant} - 2\rho\mathrm{U}^2\sin^2\theta. \quad \ldots\ldots\ldots\ldots (35)$$

Since this is unaltered when θ is replaced by $-\theta$, or by $\pm(\pi - \theta)$, it is evident that the stream exerts no resultant force on the cylinder. Some qualification of this result will be given presently. Meantime we note that if we superpose a general velocity U from right to left, we get the case of a cylinder moving with uniform velocity (and zero resistance) through a fluid which is at rest at infinity. The stream-function then has the form

$$\psi = \frac{Ua^2}{r}\sin\theta, \qquad (36)$$

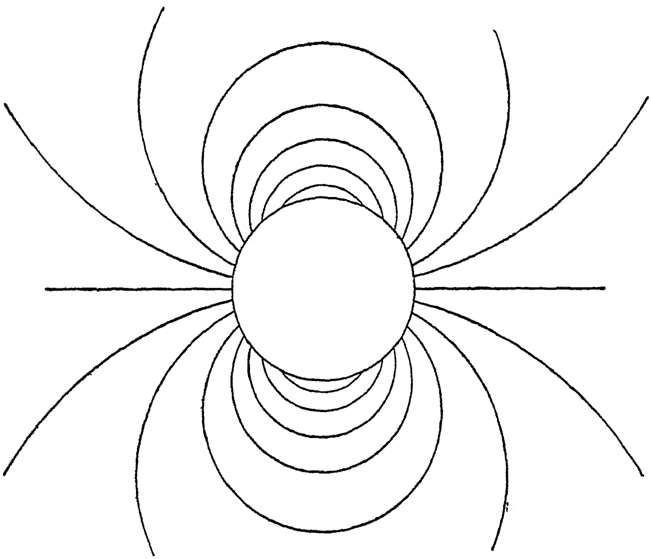


Fig. 11

so that the relative stream-lines are portions of the circles $r = C\sin\theta$, which touch the axis of x at the origin. If we calculate the square of the velocity from (36), we find

$$q^2 = \left(\frac{\partial\psi}{\partial r}\right)^2 + \left(\frac{\partial\psi}{r\partial\theta}\right)^2 = \frac{U^2a^4}{r^4}.$$

The total kinetic energy of the fluid is therefore

$$\int_a^\infty \tfrac{1}{2}\rho q^2 \cdot 2\pi r dr = \tfrac{1}{2}\pi\rho U^2a^2 = \tfrac{1}{2}M'U^2, \qquad (37)$$

where M′ is the mass of fluid displaced by the cylinder. The effect

of the presence of the fluid is therefore virtually to increase the inertia of the latter by M′.

Another simple type of motion is where the fluid moves in concentric circles about O. The velocity is then a function of r only. If the motion is irrotational we must have, by (22),

$$qr = \text{constant} = \kappa/2\pi,$$

$$\text{or} \qquad r\frac{\partial\psi}{\partial r} = \frac{\kappa}{2\pi},$$

where κ is the constant value of the circulation round O. Thus

$$\psi = \frac{\kappa}{2\pi}\log r, \qquad (38)$$

an additive constant being without effect. This corresponds to the case of a concentrated *line-vortex* at the origin, and would give infinite velocity there. For this reason (38) can only relate to cases where the origin is external to the space occupied by the fluid.

The combination of (33) and (38) makes

$$\psi = -\mathrm{U}\left(r - \frac{a^2}{r}\right)\sin\theta + \frac{\kappa}{2\pi}\log r. \qquad (39)$$

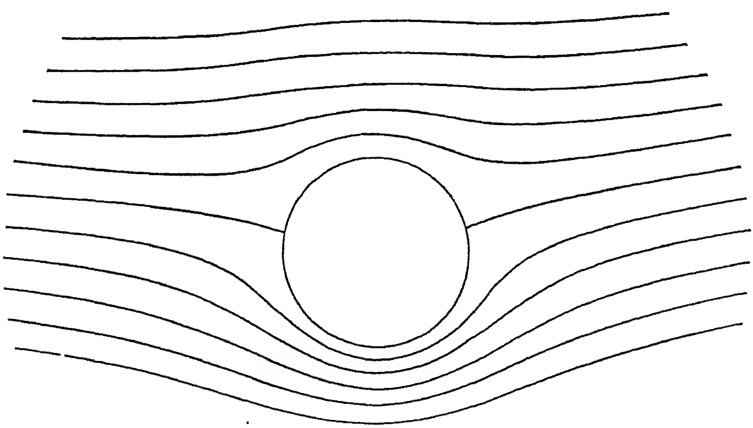


Fig. 12

The tangential velocity at the cylinder is now

$$\frac{\partial\psi}{\partial r} = -2\mathrm{U}\sin\theta + \frac{\kappa}{2\pi a},$$

whence

$$p = \text{constant} - 2\rho U^2 \sin^2\theta + \rho\frac{\kappa U}{\pi a}\sin\theta \ldots\ldots \quad (40)$$

The last term is the only one which contributes to a resultant force. Since it is the same for θ and $\pi - \theta$, there is on the whole no force parallel to Ox. The force parallel to Oy is

$$-\int_0^{2\pi} p \sin\theta a d\theta = -\frac{\rho k U}{\pi}\int_0^{2\pi}\sin^2\theta d\theta = -\rho\kappa U \ldots (41)$$

This resultant effect is due to the fact that (if κ be positive) the circulation diminishes the velocity above the cylinder and increases it below, and that a smaller velocity implies (other things being the same) a greater pressure. It may be shown that the result is the same for a cylinder of any form of section, as might be expected from the fact that it does not depend on the radius a. This theorem is the basis of Prandtl's theory of the lift of an aeroplane.

5. *Velocity-potential*

We may imagine any area occupied by fluid to be divided by a double series of lines crossing it into infinitesimal elements. The circulation round the boundary of the area will be equal to the sum of the circulations round the various elements, provided these circulations be estimated in a consistent sense. For, in this sum, a

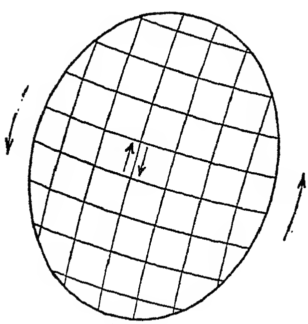


Fig. 13

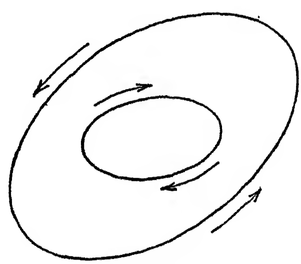


Fig. 14

side common to two adjacent elements contributes amounts which cancel. Hence if the motion be irrotational the circulation round the boundary of any area wholly occupied by fluid will be zero. We have here assumed the boundary to consist of a single closed

curve. If it consists of two such curves, what is proved is that the sum of the circulations round these in opposite senses is zero. In other words, in irrotational motion the circulation in the *same* sense is the same for any two circuits which can by continuous modification be made to coincide without passing out of the region occupied by the fluid. For example, in the case to which (38) refers, the circulation in any circuit which embraces the cylinder is κ, whilst that in any other circuit is zero.

This leads to the introduction of the function called the *velocity-potential*, in terms of which problems of irrotational motion are often discussed. This is defined by the integral

$$\phi = -\int_A^P (u\,dx + v\,dy) \quad \ldots\ldots\ldots\ldots\ldots (42)$$

taken along a line drawn from A to P. The integral has the same value for any two such lines, such as ABP, ACP in the figure, provided the space between them is fully occupied by fluid. For, reversing the direction of one of these lines, the paths ABP, PCA together form a closed circuit, round which the circulation is zero. It follows that so long as A is fixed, ϕ will be a function of the position of P only. If P′ be any point adjacent to P, the increment of ϕ in passing from P to P′ is $\delta\phi = -q_s\delta s$, where q_s is the component velocity in the direction PP′, and PP′ $= \delta s$. Hence

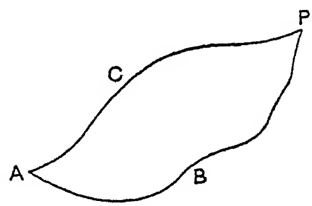


Fig. 15

$$q_s = -\frac{\partial\phi}{\partial s}, \quad \ldots\ldots\ldots\ldots\ldots\ldots (43)$$

where $\partial\phi/\partial s$ is the gradient of ϕ in the direction PP′. For instance, in rectangular co-ordinates, putting first $\delta s = \delta x$, and then $= \delta y$, we have

$$u = -\frac{\partial\phi}{\partial x}, \quad v = -\frac{\partial\phi}{\partial y} \quad \ldots\ldots\ldots\ldots (44)$$

Similarly, the radial and transverse velocities in polar co-ordinates are given by

$$u = -\frac{\partial\phi}{\partial r}, \quad v = -\frac{\partial\phi}{r\partial\theta} \quad \ldots\ldots\ldots\ldots (45)$$

From (7) and (44) we deduce

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0, \qquad (46)$$

whilst in polar co-ordinates, from (9) and (45)

$$\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right)+\frac{1}{r}\frac{\partial^2\phi}{\partial\theta^2}=0. \qquad (47)$$

It is the similarity between these relations and those met with in the theories of attractions and electrostatics that has suggested the name "velocity-potential". For the same reason the curves for which ϕ is constant are called *equipotential lines*. If in (43) δs be taken along such a line we have $q_s = 0$, showing that the equipotential lines cut the stream-lines at right angles. If on the other hand δn be the perpendicular distance between two adjacent equipotential lines, we have $\delta\phi = -q\delta n$. If, therefore, we imagine a whole system of such lines to be drawn for equal small increments $\delta\phi$, the perpendicular distance between consecutive lines will be everywhere inversely proportional to the velocity. If, further, we suppose the stream-lines to be drawn for intervals $\delta\psi$ each equal to $\delta\phi$, we have $\delta\psi = q\delta s'$, where $\delta s'$ is the interval between consecutive stream-lines of the system. Hence $\delta s' = \delta n$, showing that the stream-lines and equipotential lines drawn for equal increments of the functions will divide the region occupied by the fluid into infinitesimal squares.

The functions ϕ and ψ are connected by the relations

$$\frac{\partial\phi}{\partial x}=\frac{\partial\psi}{\partial y},\quad \frac{\partial\phi}{\partial y}=-\frac{\partial\psi}{\partial x}, \qquad (48)$$

in which the equations (25) and (46), expressing the incompressibility and the absence of vorticity, are implied. If we write

$$w=\phi+i\psi,\quad z=x+iy, \qquad (49)$$

where $i=\sqrt{(-1)}$, the relations (48) are satisfied by any assumption of the form

$$w=f(z). \qquad (50)$$

For this makes

$$\frac{\partial w}{\partial y}=if'(z)=i\frac{\partial w}{\partial x}, \qquad (51)$$

whence, substituting the value of w, and equating separately real and imaginary parts, we reproduce (48).

For example, if $w = -\mathrm{U}z$, we have

$$\phi = -\mathrm{U}x, \quad \psi = -\mathrm{U}y, \ldots\ldots\ldots\ldots (52)$$

expressing a uniform flow parallel to Ox. Again, if $w = \mathrm{C}/z$

$$\phi + i\psi = \frac{\mathrm{C}}{x + iy} = \frac{\mathrm{C}}{r}(\cos\theta - i\sin\theta) \ldots\ldots\ldots (53)$$

This corresponds to (36), if $\mathrm{C} = -\mathrm{U}a^2$, and shows that in the case referred to

$$\phi = -\frac{\mathrm{U}a^2}{r}\cos\theta \ldots\ldots\ldots\ldots\ldots (54)$$

A more general assumption is

$$w = \mathrm{C}z^n,$$

or $\quad \phi + i\psi = \mathrm{C}(x + iy)^n = \mathrm{C}r^n(\cos n\theta + i\sin n\theta).$

The stream-function is now

$$\psi = \mathrm{C}r^n \sin n\theta,$$

which vanishes both for $\theta = 0$ and $\theta = \alpha$, provided $n = \pi/\alpha$. Taking these lines as fixed boundaries we have the flow in an angle, or round a salient, according as $\alpha \lessgtr \pi$. The radial and transverse velocities are, by (8),

$$-n\mathrm{C}r^{n-1}\cos n\theta \text{ and } n\mathrm{C}r^{n-1}\sin n\theta,$$

respectively. If $\alpha < \pi, n > 1$, and these expressions vanish at the vertex where $r = 0$. If, on the other hand $\alpha > \pi$, $n < 1$, and the velocity there is infinite. Even if the salient be rounded off, the velocity may be very great, with the result that the pressure falls much below the value at a distance. It is otherwise obvious that if the fluid is to be guided round a sharp curve there must be a rapid increase of pressure outwards to balance the centrifugal force. If this is not sufficient a vacuum is formed and "cavitation" ensues.

If $w = \mathrm{C}\log z$, where C is real,

$$\phi + i\psi = \mathrm{C}\log(x + iy) = \mathrm{C}\log re^{i\theta} = \mathrm{C}\log r + i\mathrm{C}\theta. \quad (55)$$

This represents a line-source of strength m, if to agree with (29) we put $\mathrm{C} = -m/2\pi$. The corresponding value of ϕ is

$$\phi = -\frac{m}{2\pi}\log r. \ldots\ldots\ldots\ldots\ldots (56)$$

If on the other hand C is a pure imaginary, $= i\mathrm{A}$, say,

$$\phi + i\psi = -\mathrm{A}\theta + i\mathrm{A}\log r \qquad (57)$$

This represents the case of the line-vortex to which (38) refers, if we put $\mathrm{A} = \kappa/2\pi$, and so make

$$\phi = -\frac{\kappa}{2\kappa}\theta \qquad (58)$$

The function ϕ has an important dynamical interpretation. Any state of motion in which there is no circulation in any circuit, and in which, therefore, ϕ has a definite value at every point, could be generated instantaneously from rest by a proper application of impulsive pressures over the boundary. For the requisite condition for this is that the resultant of the impulsive pressures (ϖ) on the surface of any small volume Q should be equivalent to the momentum acquired by this. Hence if q_s is the component velocity in the direction of any linear element δs we must have

$$-\frac{\partial\varpi}{\partial s}\mathrm{Q} = \rho\mathrm{Q}q_s = -\rho\mathrm{Q}\frac{\partial\phi}{\partial s},$$

which is satisfied if

$$\varpi = \rho\phi \qquad (59)$$

Hence ϕ determines the impulsive pressure requisite to start the actual motion in the above manner.

As an example, we may take the case of a cylinder moving through a large mass of liquid, without circulation, to which the formula (54) refers. The resultant of the impulsive pressures on the surface of the cylinder is parallel to Ox, of amount

$$-\int_0^{2\pi} \omega \cos\theta \, a d\theta = \rho \mathrm{U} a^2 \int_0^{2\pi} \cos^2\theta \, d\theta = \mathrm{M'U}, \qquad (60)$$

if $\mathrm{M'} = \pi\rho a^2$ as before. The total impulse which must be given to the *cylinder* to start the motion is therefore $(\mathrm{M} + \mathrm{M'})\mathrm{U}$. This confirms the former result that the inertia of the cylinder is virtually increased by the amount M′.

6. *Motion with Axial Symmetry. Sources and Sinks*

The second type of motion to which reference was made on p. 54 is where the flow takes place in a system of planes passing through an axis, which we take as axis of x, and is the same in each such plane. We denote by x, y the co-ordinates in one of these planes, by r distance from the origin, and by θ the angle which r makes with Ox. The conditions for steady motion are obtained by the previous process. Resolving along a stream-line we should be led to Bernoulli's equation (2); whilst the normal resolution in an axial plane yields equations of the same form as (15) and (17), provided ζ now denotes the vorticity in that plane. The inference as to the distribution of vorticity is however altered. The space between two consecutive stream-lines now represents a section of a thin shell, of revolution about Ox, and the flux in this is accordingly $q \cdot 2\pi y \delta n$. Comparing with (18), we see that along any stream-line ζ must vary as y. We may conceive the fluid as made up of annular filaments having Ox as a common axis. The section of such a filament, as it moves along, will vary inversely as y; hence the product of the vorticity into the cross-section must remain constant. This is a particular case of a general theorem that the *strength* of a *vortex-filament* (in this case a *vortex-ring*) remains unaltered as it moves.

If $\zeta = 0$, the argument for the existence of a velocity-potential will hold as before. One or two simple cases may be noticed. If we imagine a *point-source* at O, the flux outwards across any concentric spherical surface of radius r must be equal to the output (m) per unit time whence

$$-\frac{\partial \phi}{\partial r}.4\pi r^2 = m, \text{ or } \phi = \frac{m}{4\pi}.\frac{1}{r}. \quad \ldots\ldots\ldots(61)$$

We may apply this solution to the collapse of a spherical bubble. If R be the radius at time t, we have

$$\phi = \frac{R^2}{r}\frac{dR}{dt}, \quad \ldots\ldots\ldots\ldots\ldots\ldots(62)$$

since this makes $-\partial\phi/\partial r = dR/dt$ for $r = R$. The corresponding kinetic energy of the fluid is

$$\int_R^\infty \tfrac{1}{2}\rho q^2.4\pi r^2 dr = 2\pi\rho R^4\left(\frac{dR}{dt}\right)^2\int_R^\infty \frac{dr}{r^2} = 2\pi\rho R^3\left(\frac{dR}{dt}\right)^2. \ldots.(63)$$

If p_0 be the pressure at a distance, the rate at which work is being done on the fluid enclosed in a spherical surface of large radius r is

$$-p_0 q \cdot 4\pi r^2 = -4\pi p_0 R^2 \frac{dR}{dt}, \qquad (64)$$

the pressure inside the bubble being neglected. Equating the rate of increase of the energy to the work done,

$$\frac{d}{dt} R^3 \left(\frac{dR}{dt}\right)^2 = -\frac{2p_0}{\rho} R^2 \frac{dR}{dt}, \qquad (65)$$

whence

$$R^3 \left(\frac{dR}{dt}\right)^2 = \tfrac{2}{3}\frac{p_0}{\rho}(R_0^3 - R^3), \qquad (66)$$

where R_0 is the initial radius of the cavity. It is not easy to integrate this further in a practical form, but the time of collapse happens to be ascertainable; it is

$$\tau = 0{\cdot}915 R_0 \sqrt{\frac{\rho}{p_0}}. \qquad (67)$$

Thus if p_0 be the atmospheric pressure, and $R_0 = 1$ cm., τ is less than the thousandth part of a second. The total energy lost, or rather converted into other forms is, from (63) and (66),

$$\frac{4}{3}\pi p_0 R_0^3. \qquad (68)$$

In the particular case referred to, this is $4{\cdot}19 \times 10^6$ ergs, or $0{\cdot}312$ of a foot-pound.

The expansion of a spherical cavity owing to the pressure of an included gas can be treated in a similar way. This illustrates, at all events qualitatively, the early stages of a submarine explosion. The potential energy of a gas compressed under the adiabatic condition to volume v and pressure p is $pv/(\gamma - 1)$, where γ is the ratio of the two specific heats. If p be the internal pressure when the radius of the cavity is R, and p_0 its initial value, we have by the adiabatic law

$$\frac{p}{p_0} = \left(\frac{R_0}{R}\right)^{3\gamma}. \qquad (69)$$

The potential energy is therefore

$$\frac{4p_0 R_0^3}{3(\gamma - 1)}\left(\frac{R_0}{R}\right)^{3\gamma - 3}. \qquad (70)$$

Expressing that the total energy is constant, we have

$$2\pi\rho R^3\left(\frac{dR}{dt}\right)^2 = \frac{4p_0R_0^3}{3(\gamma - 1)}\left\{1 - \left(\frac{R_0}{R}\right)^{3\gamma-3}\right\},$$

or

$$\left(\frac{dR}{dt}\right)^2 = \frac{2c_0^2}{3(\gamma - 1)}\left\{\left(\frac{R_0}{R}\right)^3 - \left(\frac{R_0}{R}\right)^{3\gamma}\right\}, \quad\ldots\ldots\ldots(71)$$

where

$$c_0 = \sqrt{(p_0/\rho)}. \quad\ldots\ldots\ldots\ldots\ldots\ldots(72)$$

This quantity c_0, which is of the dimensions of a velocity, is a measure of the rapidity with which the changes take place. It is not easy to carry the solution further except in the particular case of $\gamma = \frac{4}{3}$. If we write

$$R/R_0 = 1 + z, \quad\ldots\ldots\ldots\ldots\ldots\ldots(73)$$

we have then

$$(1 + z)^2\frac{dz}{dt} = \frac{c_0}{R_0}\sqrt{(2z)}, \quad\ldots\ldots\ldots\ldots\ldots(74)$$

whence

$$\frac{c_0t}{R_0} = \sqrt{(2z)}\,(1 + \tfrac{2}{3}z + \tfrac{1}{5}z^2). \quad\ldots\ldots\ldots\ldots(75)$$

This gives the time taken by the radius of the cavity to attain any assigned value R. The following table gives a few examples.

R/R_0.	c_0t/R_0.
1	0
2	2·64
3	6·27
4	11·76
5	19·42

As a concrete illustration, suppose the initial diameter of the cavity to be 1 m., and the initial pressure p_0 to be 1000 atmospheres, so that $c_0 = 3{\cdot}16 \times 10^4$ cm./sec. We find that the radius is doubled in $\frac{1}{250}$ sec., and multiplied five-fold in about $\frac{1}{30}$ sec. It must be remembered that in this investigation, as in the preceding one, the water has been assumed to be incompressible. With an initial internal pressure of the order of 10,000 atmospheres, we obtain values of dR/dt comparable with the velocity of sound in water. The influence of compressibility then ceases to be negligible.

The combination of a source m at a point A and a corresponding sink $-m$ at B gives

$$\phi = \frac{m}{4\pi}\left(\frac{1}{r_1} - \frac{1}{r_2}\right). \qquad (76)$$

If we imagine the points A and B to approach one another, whilst the product mBA is constant $(= \mu)$, we have ultimately $r_2 - r_1 =$ AB $\cos\theta$, and

$$\phi = \frac{\mu}{4\pi}\,\frac{\cos\theta}{r^2}. \qquad (77)$$

We have here the conception of a *double-source*. If we combine this with a uniform flow $\phi = -\mathrm{U}x = -\mathrm{U}r\cos\theta$ parallel to Ox. we have

$$\phi = -\left(\mathrm{U}r - \frac{\mu}{4\pi r^2}\right)\cos\theta.$$

This makes $-\partial\phi/\partial r = 0$ for $r = a$, provided $\mu = -2\pi\mathrm{U}a^3$ The formula

$$\phi = -\mathrm{U}\left(r + \frac{a^3}{2r^2}\right)\cos\theta. \qquad (78)$$

therefore gives the steady flow past a sphere of radius a. The tangential velocity at the surface is

$$-\frac{\partial\phi}{r\partial\theta} = -\tfrac{3}{2}\mathrm{U}\sin\theta,$$

and the pressure is accordingly

$$p = \text{constant} - \tfrac{9}{8}\rho\mathrm{U}^2\sin^2\theta. \qquad (79)$$

Since this is the same when θ is replaced by $\pi - \theta$, the resultant effect on the sphere is nil. If we superpose a general velocity $-\mathrm{U}$, we get the case where the sphere is in motion with velocity U in the negative direction of x; thus

$$\phi = -\frac{\mathrm{U}a^3}{2r^2}\cos\theta. \qquad (80)$$

If we imagine this motion to be produced instantaneously from rest, the impulsive pressure of the fluid on the sphere, in the direction of x-negative, is

$$\int_0^{\mathrm{R}} \varpi\cos\theta . 2\pi a\sin\theta a d\theta = \int_0^{\pi} \rho\phi\cos\theta . 2\pi a\sin\theta a d\theta$$
$$= -\tfrac{2}{3}\pi\rho a^3\mathrm{U}, \qquad (81)$$

or $-\frac{1}{2}M'U$, where M' is the mass of fluid displaced. The impulse which must be given to the sphere to counteract this is $\frac{1}{2}M'U$, and the total impulse in the direction of the velocity is $(M + \frac{1}{2}M')U$, where M is the mass of the sphere itself. It is a proposition in Dynamics that the kinetic energy due to a system of impulses is got by multiplying each constituent of the impulse by the velocity produced in its direction, and taking half the sum of such products. In the present case this gives $\frac{1}{2}(M + \frac{1}{2}M')U^2$. The case is analogous to that of the cylinder, already treated, except that the virtual addition to the mass is $\frac{1}{2}M'$ instead of M'.

This result, viz. that the effect of a *frictionless* liquid on a body moving through it without rotation consists merely in an addition to its inertia, is quite general. Whatever the form of the body, the impulsive pressure necessary to start the actual motion of the fluid instantaneously from rest will evidently be proportional to the velocity (U), and the reaction on the body in the direction of motion will therefore be $-kM'U$, where k is some numerical coefficient. The impulse necessary to be given to the solid is therefore $(M + kM')U$. A similar conclusion would follow from the consideration of the energy produced. The value of k will, of course, depend on the form of the solid and the direction of its motion. The following table gives values for the case of a prolate ellipsoid, the ratio c/a being that of the longer to the shorter semi-diameter. The column under "k_1" relates to motion "end-on", and that under "k_2" to motion "broadside-on".

c/a.	k_1.	k_2.
1 (sphere)	$\frac{1}{2}$	$\frac{1}{2}$
1·5	0·305	0·621
2·0	0·209	0·702
3·0	0·122	0·803
4·0	0·082	0·860
5·0	0·059	0·895
6·0	0·045	0·918
7·0	0·036	0·933
8·0	0·029	0·945
9·0	0·024	0·954
10·0	0·021	0·960
∞ (cylinder)	0	1

Any line AP drawn in a plane through the axis represents an

annular portion of a surface of revolution about Ox. The flux across this portion, say from right to left, will be the same for any two lines from A to P, provided the space between them is occupied by fluid. If A be fixed, this flux will therefore be a function only of the position of P; we denote it by $2\pi\psi$. If PP$'$ be a linear element δs, drawn in any direction, the flux across the surface generated by its revolution about Ox will be

$$2\pi\delta\psi = q_n . 2\pi y \delta s,$$

where q_n is the velocity normal to δs. Hence

$$q_n = \frac{1}{y}\frac{\partial\psi}{\partial s} \quad\quad (82)$$

It was to simplify this formula that the factor 2π was introduced in the definition of ψ. As particular cases of (79), the component velocities parallel and perpendicular to Ox are

$$u = -\frac{1}{y}\frac{\partial\psi}{\partial y}, \quad v = \frac{1}{y}\frac{\partial\psi}{\partial x} \quad\quad (83)$$

The lines for which ψ is constant are stream-lines, and ψ is called the *stream-function.*

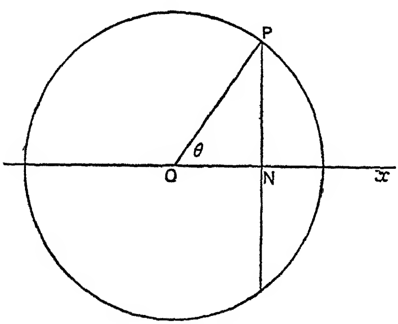


Fig. 16

To find ψ for the case of a point-source, we calculate the flux across the segment of a spherical surface, with OP as radius, cut off by a plane through P perpendicular to Ox. The radial velocity across this segment is $m/4\pi r^2$, and the area is $2\pi r(r - x)$, where r = OP, x = ON. Hence, omitting an additive constant, the flux in the desired sense is

$$2\pi\psi = \tfrac{1}{2}mx/r, \text{ or } \psi = \frac{m}{4\pi}\cos\theta \quad\quad (84)$$

The combination of an equal source and sink at A and B gives

$$\psi = \frac{m}{4\pi}(\cos\theta_1 - \cos\theta_2), \quad\quad (85)$$

whilst if A and B are made to approach coincidence in such a way that $m\mathrm{AB} = \mu$, we have ultimately

$$\delta\,(\cos\theta) = -\sin\theta\,\delta\theta = -\sin\theta\,(\mathrm{AB}\sin\theta)/r,$$

and therefore
$$\psi = -\frac{\mu}{4\pi}\frac{\sin^2\theta}{r}.\ldots\ldots\ldots\ldots\ldots\ldots(86)$$

For a uniform flow parallel to Ox, we have $2\psi = -\mathrm{U}y^2$, and if we superpose this on (85) or (86) we get stream-line forms, one of which may be taken as the profile of a stationary solid in the stream. For instance, combining with (86), and putting $\mu = -2\pi\mathrm{U}a^3$,

$$\psi = -\frac{\mathrm{U}}{2}\left(r^2 - \frac{a^3}{r}\right)\sin^2\theta.\ldots\ldots\ldots\ldots(87)$$

The line $\psi = 0$ now consists of the circle $r = a$ and of the portions of the axis of x external to it. If we now remove the uniform flow we get the lines of motion due to the sphere moving in the direction of x-negative with velocity U.

The process just indicated admits of great extension. By taking a series of sources and sinks, not necessarily concentrated in points, along the axis of x, subject to the proviso that the aggregate output is zero, and superposing a uniform flow, we may obtain a variety of curves which may serve as the profile of a moving solid. This procedure was originated by Rankine from the point of view of naval architecture, and has recently been applied to devise profiles which imitate those of airships. Since the motion of the fluid is known, the pressure distribution over the surface can be calculated and compared with model experiments.

7. *Tracing of Stream-lines*

There are various methods by which drawings of systems of stream-lines can be constructed. For example, suppose that the stream-function consists of two parts ψ_1, ψ_2, which are themselves readily traced. Drawing the curves

$$\psi_1 = m\alpha, \quad \psi_2 = n\alpha,$$

where m, n are integers, and α is some convenient constant (the smaller the better), these will divide the plane of the drawing into (curvilinear) quadrilaterals. The curves

$$\psi_1 \pm \psi_2 = n\alpha$$

will form the diagonals of these quadrilaterals, and are accordingly easily traced if the compartments are small enough. For instance, in the case of (33), where we may put $U = -1$, $a = 1$, without any effect except on the scale of the diagram, we should trace the straight lines

$$r \sin\theta = ma,$$

which are parallel to the axis of x and equidistant, and the circles

$$r = \frac{1}{na} \sin\theta.$$

Another method is to write the equation (as above modified) in the form

$$\psi = y\left(1 - \frac{1}{r^2}\right),$$

and to tabulate the function $1/(1 - 1/r^2)$ for a series of equidistant values of r, beginning with unity. This is easily done with the help of Barlow's tables. The values of y where a particular stream-line crosses the corresponding circles are then given by

$$y = \frac{\psi}{1 - \frac{1}{r^2}}.$$

Giving ψ in succession such values as 0·1, 0·2, 0·3, ... a system of stream-lines is readily drawn. The same numerical work comes in useful in the case of (39). A similar process can be applied to tracing the stream-lines past a sphere, to which (87) refers.

A more difficult example is presented by equation (99) later. Nothing is altered except the scale if we write this in the form

$$\psi = \log\frac{r_1^2}{r_2^2} - x,$$

whence

$$\frac{(x + 1)^2 + y^2}{(x - 1)^2 + y^2} = e^{x+\psi},$$

and therefore

$$r^2 + 1 = 2x\frac{e^{x+\psi} + 1}{e^{x+\psi} - 1} = 2x \coth\frac{x + \psi}{2}.$$

The hyperbolic function on the right-hand has been tabulated, so that we can calculate the values of r (the distance from the origin)

at the points where any given stream-line curve cuts the lines x = constant.

8. *General Equations of Motion*

The general equations of hydrodynamics have so far not been required. To obtain them in their full three-dimensional form we denote by u, v, w the component velocities parallel to rectangular axes at the point (x, y, z) at the time t. They are therefore functions of the four independent variables x, y, z, t. If we fix our attention on a particular instant t_0, their values would gives us a picture of the instantaneous state of motion throughout the field. If on the other hand we fix our attention on a particular point (x_0, y_0, z_0) in the field, their values as functions of t would give us the history of what takes place at that chosen point. We introduce a symbol D/Dt to denote a differentiation of any property of the fluid considered as belonging to a particular particle. Thus Du/Dt denotes the component acceleration of a particle parallel to Ox; this is to be distinguished from $\partial u/\partial t$, which is the rate at which u varies at a particular place. The dynamical equations are obtained by equating the rate of change of momentum of a given small portion of the fluid to the forces acting on it. Considering the portion which at time t occupies a rectangular element $\delta x\delta y\delta z$, we have, resolving parallel to Ox

$$\rho\delta x\delta y\delta z\frac{Du}{Dt} = -\frac{\partial p}{\partial x}\delta x\delta y\delta z - \rho\delta x\delta y\delta z\frac{\partial V}{\partial x},$$

where the first term on the right hand is the effect of the fluid pressures on the boundary of the element, as determined by (1), whilst the second term is due to extraneous forces (such, for example, as gravity) which are supposed to be conservative, V being the potential energy per unit mass. Thus we find,

$$\left.\begin{aligned}\rho\frac{Du}{Dt} &= -\frac{\partial p}{\partial x} - \rho\frac{\partial V}{\partial x},\\ \rho\frac{Dv}{Dt} &= -\frac{\partial p}{\partial y} - \rho\frac{\partial V}{\partial y},\\ \rho\frac{Dw}{Dt} &= -\frac{\partial p}{\partial z} - \rho\frac{\partial V}{\partial z}.\end{aligned}\right\}\qquad\qquad(88)$$

To find expressions for Du/Dt, &c., let P, P′ be the positions occupied by a particle at two successive instants t_1, t_2. Let u_1, u_1'

be the values of u at the points P, P′, respectively, at time t_1, and u_2, u_2' the corresponding values at time t_2. The average acceleration of the particle parallel to Ox in the interval $t_2 - t_1$ is therefore

$$\frac{u_2' - u_1}{t_2 - t_1} = \frac{u_2 - u_1}{t_2 - t_1} + \frac{u_2' - u_2}{t_2 - t_1}.$$

The limiting value of the left-hand side is Du/Dt; that of the first term on the right is $\partial u/\partial t$, the rate of change of u at P. Again, $u_2' - u_2$ is the difference of simultaneous component velocities at the points P′ and P, so that

$$u_2' - u_2 = \frac{\partial u}{\partial s}.\mathrm{PP}' = \frac{\partial u}{\partial s}q(t_2 - t_1),$$

where q is the resultant velocity

$$\sqrt{(u^2 + v^2 + w^2)},$$

and $\partial u/\partial s$ is the gradient of u in the direction PP′. Thus

$$\frac{\mathrm{D}u}{\mathrm{D}t} = \frac{\partial u}{\partial t} + q\frac{\partial u}{\partial s}. \quad\quad (89)$$

Now if l, m, n be the direction cosines of δs,

$$\frac{\partial u}{\partial s} = l\frac{\partial u}{\partial x} + m\frac{\partial u}{\partial y} + n\frac{\partial u}{\partial z},$$

and $lq = u, mq = v, nw = w$. Hence, finally,

$$\frac{\mathrm{D}u}{\mathrm{D}t} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}. \quad\quad (90)$$

Similar expressions are obtained for Dv/Dt, Dw/Dt. Substituting in (88) we get the dynamical equations in their classical form.

To these must be added a kinematical relation, which expresses that the total flux outwards across the boundary of the element $\delta x\delta y\delta z$ is zero. The two faces perpendicular to Ox give $- u\delta y\delta z$, and $(u + \frac{\partial u}{\partial x}\delta x)\delta y\delta z$ respectively, the sum of which is $\partial u/\partial x\delta x\delta y\delta z$. Taking account in like manner of the flux across the remaining faces, and equating the total to zero, we have the equation of continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \quad\quad (91)$$

of which (7) is a particular case.

When the motion is irrotational we have

$$\frac{\partial u}{\partial y} = -\frac{\partial^2 \phi}{\partial y \partial x} = \frac{\partial v}{\partial x},$$

and similar relations, so that (90) becomes

$$\frac{\mathrm{D}u}{\mathrm{D}t} = -\frac{\partial^2 \phi}{\partial x \partial t} + \frac{1}{2}\frac{\partial q^2}{\partial x}, \quad \text{............(92)}$$

When this is substituted in (88), it is seen that the dynamical equations have the integral

$$\frac{p}{\rho} = \frac{\partial \phi}{\partial t} - \tfrac{1}{2}q^2 - \mathrm{V} + \mathrm{F}(t), \quad \text{..........(93)}$$

where $\mathrm{F}(t)$ denotes a function of t only which is to be determined by the boundary conditions, but has no effect on the motion. It is evident beforehand that a pressure uniform throughout the liquid, even if it varies with the time, is without effect. The occurrence of $\mathrm{F}(t)$ in the present case is merely a consequence of the fact already mentioned that in an absolutely incompressible fluid changes of pressure are transmitted instantaneously.

The equation of continuity (91) now takes the form

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0. \quad \text{.............(94)}$$

In steady motion $\partial\phi/\partial t = 0$, and (93) reduces to our former result (27).

Vortex Motion

1. *Persistence of Vortices*

Turning now to the consideration of vortex motion, the fundamental theorem in the subject is that the circulation in any circuit moving with the fluid (i.e. one which consists always of the same particles) does not alter with the time. For, consider any element $u\delta x$ of the integral

$$\int (udx + vdy + wdz),$$

which expresses the circulation. We have

$$\frac{\mathrm{D}}{\mathrm{D}t}(u\delta x) = \frac{\mathrm{D}u}{\mathrm{D}t}\delta x + u\frac{\mathrm{D}(\delta x)}{\mathrm{D}t} \quad \text{...........(95)}$$

Now $D(\delta x)/Dt$ is the rate at which the projection on the axis of x, of the line joining two adjacent particles, is increasing, and is therefore equal to δu. Hence,

$$\frac{D}{Dt}(u\delta x) = \frac{Du}{Dt}\delta x + u\delta u = -\frac{1}{\rho}\frac{\partial p}{\partial x}\delta x - \frac{\partial V}{\partial x}\delta x + \tfrac{1}{2}\delta(u^2),$$

and therefore

$$\frac{D}{Dt}(u\delta x + v\delta y + w\delta z) = -\delta\left(\frac{p}{\rho} + V - \tfrac{1}{2}q^2\right). \ldots .(96)$$

When this is integrated round the circuit, the result is zero. Hence

$$\frac{D}{Dt}\int(udx + vdy + wdz) = 0 \ldots\ldots\ldots\ldots(97)$$

It is important to notice the restrictions under which this is proved. It is assumed that the density is uniform, that the fluid is frictionless, and that the external forces have a potential. The first of these assumptions is violated, for instance, when convection currents are produced by unequal heating of a mass of water, owing to the variation of density. The second assumption fails when the influence of viscosity becomes sensible.

Irrotational motion is characterized by the property that the circulation is zero in every infinitesimal circuit. We now have a general proof that if this holds for a particular portion of fluid at any one instant, it will (under the conditions stated) continue to hold for that particular portion, whether there be rotational motion in other parts of the mass or not. Again, in two-dimensional motion we have seen that the circulation round any small area is equal to the product of the vorticity ζ into the area. Since the area occupied by any portion of fluid remains constant as it moves along, we infer that the vorticity also is constant. This has already been proved otherwise in the case of steady motion. The value of ζ is, of course, constant along a line drawn normal to the planes of motion. Such a line is a *vortex-line* according to a general definition to be given presently, and the vortex-lines passing through any small contour enclose what is called a *vortex-filament*, or simply a *vortex*. The *strength* of a vortex is defined by the product of the vorticity into the cross-section, i.e. by the circulation immediately round it.

Still keeping for the moment to the case of two-dimensions; we have seen that the circulation round the boundary of any area

occupied by the fluid is equal to the sum of the circulations round the various elements into which it may be divided, provided these be estimated in a consistent sense. In virtue of the above definitions an equivalent statement is that the circulation in any circuit is equal to the sum of the strengths of all the vortices which it embraces.

2. *Isolated Vortices*

The stream- and velocity-functions due to an isolated rectilinear vortex of strength κ have already been met with in (38) and (58). The velocity distributions due to two or more parallel rectilinear vortices may be superposed. Suppose, for instance, we have a *vortex-pair* composed of two vortices A, B of equal and opposite strengths $\pm\,\kappa$.

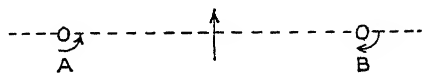


Fig. 17

Each produces in the other a velocity $\kappa/2\pi a$, where a is the distance apart, at right angles to AB. The pair advances therefore with this constant velocity, the distance apart being unaltered. The lines of flow are given by

$$\psi = \frac{\kappa}{2\pi}\log\frac{r_1}{r_2}, \qquad (98)$$

where r_1, r_2 are the distances of A, B respectively from the point P to which ψ refers. The lines for which the ratio r_1/r_2 has the same value are co-axial circles having A, B as limiting points. If we superpose a uniform flow $\kappa/2\pi a$ in the direction of y negative, the case is reduced to one of steady motion, and the stream-function is now

$$\psi = \frac{\kappa}{2\pi}\left(\log\frac{r_1}{r_2} - \frac{x}{a}\right) \qquad (99)$$

The stream-line $\psi = 0$ consists partly of the axis of y, where $r_1 = r_2$ and $x = 0$, and partly of a closed curve which surrounds always the same mass of the fluid. This portion is carried forward by the vortex-pair in the original form of the problem.

If a flat blade, e.g. a paper-knife, held vertically, be dipped into water, and moved at right angles to its breadth for a short distance, and then rapidly withdrawn, a vortex-pair will be produced by friction at the edges, and will be seen to advance in accordance with the preceding theory. The positions of the vortices are marked by

the dimples produced on the water surface. In this way the action of one vortex-pair on another may be studied.

The detailed study of vortex motion in three dimensions would lead us too far, but a brief sketch of the fundamental relations may be given. It is necessary in the first place to introduce the notion of vorticity as a vector. Through any point P we draw three lines PA, PB, PC parallel to the co-ordinate axes, meeting any plane drawn infinitely near to P in the points A, B, C. It is evident at once from the figure that the circulation round ABC is equal to the sum of the circulation round the triangles PBC, PCA, PAB, provided the positive direction of the circulations be right-handed as regards the positive directions of the co-ordinate axes. Now, if

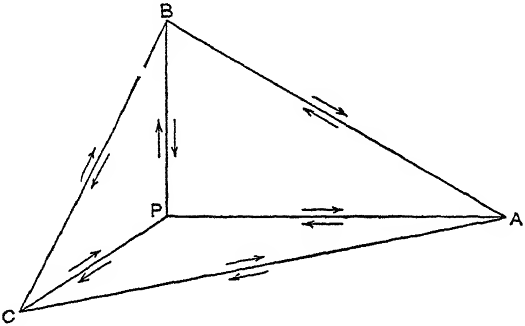


Fig. 18

l, m, n be the direction-cosines of the normal drawn from P to the plane ABC, and Δ the area ABC, the areas of the above triangles are $l\Delta$, $m\Delta$, $n\Delta$, respectively. Hence if ξ, η, ζ be the vorticities in these planes, i.e. the ratios of these circulations to the respective areas, the circulation round ABC will be

$$(l\xi + m\eta + n\zeta)\Delta \ldots\ldots\ldots\ldots\ldots (100)$$

We may regard ξ, η, ζ as the components of a vector ω, and the expression (100) is then equal to $\omega \cos\theta\, \Delta$ where θ is the angle which the normal to Δ makes with the direction of ω. In other words, the vorticity in any plane is equal to the component of ω along the normal to that plane.

The value of ζ has been given in (12). Writing down the corresponding formulæ for ξ and η, we have altogether

$$\xi = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \quad \eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \quad \zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \ldots (101)$$

We have, of course,

$$\omega^2 = \xi^2 + \eta^2 + \zeta^2 \qquad (102)$$

A line drawn from point to point always in the direction of the vector (ξ, η, ζ) is called a *vortex-line*. The vortex-lines which meet any given curve generate a surface such that the circulation in every circuit drawn on it is zero. If the curve in question be closed, and infinitely small, the fluid enclosed by the surface constitutes a *vortex-filament*, or simply a *vortex*. Consider a circuit such as ABCAA′C′B′A′A in the figure, drawn on the wall of the filament. Since the circulation in it is zero, and since the portions due to AA′ and A′A cancel, the circulation round ABC is equal to that round A′B′C′. Supposing the planes of these two curves to be cross-sections of the filament, we learn that the product of the resultant vorticity into the cross-section has the same value along the vortex. This product is called the *strength* of the vortex. The dynamical theorem above proved shows that under the conditions postulated the strength of a vortex does not vary with the time. The constancy of the strength of a *vortex-ring* has already been proved in the case of steady motion.

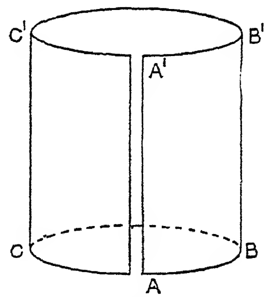


Fig. 19

The argument by which the circulation in a plane circuit was, under a certain condition, proved to be equal to the sum of the strengths of all the vortices which it embraces, is easily extended (under a similar condition) to the general case.

The most familiar instance of isolated vortices is that of smoke rings, which are generated in the first instance by viscosity, but retain a certain degree of persistence. A vortex-ring at a distance from other vortices, or from the boundaries of the fluid, advances along its axis with uniform velocity. The mutual influence of vortex-rings is closely analogous to that of vortex-pairs.

Wave Motion

1. *Canal Waves*

Water-waves are by no means the simplest type of wave-motion met with in Mechanics, and the general theory is necessarily somewhat intricate, even when we restrict ourselves to oscillations of

small amplitude. The only exception is in the case of what are variously called *long waves*, or *tidal waves*, or *canal waves*, the characteristic feature being that the wave-length is long compared with the depth, and the velocity of the fluid particles therefore sensibly uniform from top to bottom.

Taking this case first, we inquire under what condition a wave can travel without change of form, and therefore with a definite velocity. Supposing this velocity to be c, from left to right, we may superpose a general velocity $-c$ in the opposite direction and so reduce the problem to one of steady motion. The theory is now the same as for the flow through a pipe of gradually varying section, except that the upper boundary is now a free surface, instead of a rigid wall. If h be the original depth, the velocity where the surface-elevation is η will be

$$q = -\frac{ch}{h+\eta}. \qquad \qquad (103)$$

The pressure along the wave-profile, which is now a stream-line, is given by Bernoulli's equation

$$\frac{p}{\rho} = \text{constant} - \tfrac{1}{2}q^2 - g\eta = \text{constant} - \tfrac{1}{2}c^2\left(1+\frac{\eta}{h}\right)^{-2} - g\eta$$
$$= \text{constant} - \tfrac{1}{2}c^2\left(1-\frac{2\eta}{h}\right) - g\eta, \qquad \qquad (104)$$

approximately, if we neglect the square of η/h. This pressure will be independent of η, provided

$$c = \surd(gh). \qquad \qquad (105)$$

The required wave-velocity is therefore that which would be acquired by a particle falling vertically under gravity, from rest, through a space equal to half the depth.

If we now restore the original form of the problem, by imposing a velocity c in the positive direction, we have

$$q = c - \frac{ch}{h+\eta} = c\frac{\eta}{h}, \qquad \qquad (106)$$

approximately. The velocity of the water itself is therefore forward or backward, according as η is positive or negative, i.e. it is forward where there is an elevation, and backward where there is a depression. The potential energy per unit area of the surface is $\tfrac{1}{2}g\rho\eta^2$, and the corresponding kinetic energy is $\tfrac{1}{2}\rho q^2 h = \tfrac{1}{2}\rho c^2\eta^2/h$. Since these are

equal by (106), the energy of a progressive wave is half-potential and half-kinetic.

The condition for permanence of form has not, of course, been exactly fulfilled in the above calculation. A closer approximation to fact is evidently obtained if in (105) we replace h by $h + \eta$; this will give us the velocity of the wave-form relative to the water in the neighbourhood, which is itself moving with the velocity given by (106), if η/h is small. The elevation η is therefore propagated *in space* with the velocity

$$\sqrt{\{g(h+\eta)\}} + \sqrt{(gh)}\frac{\eta}{h} = \sqrt{(gh)}\left(1 + \frac{3}{2}\frac{\eta}{h}\right), \quad \ldots\ldots(107)$$

approximately. The more elevated portions therefore move the faster, with the result that the profile of an elevation tends to become steeper in front and more gradual in slope behind.

2. *Deep-water Waves*

Proceeding to the more general case, we will assume that the motion takes place in a series of parallel vertical planes, and is the same in each of these, so that the ridges and furrows are rectilinear. Fixing our attention on one of these planes, we take rectangular axes Ox, Oy, the former being horizontal, and the later vertical with the positive direction upwards. The problem being reduced to one of steady motion as before, the stream-function will be

$$\psi = cy + \psi_1, \quad \ldots\ldots\ldots\ldots\ldots(108)$$

where ψ_1 is supposed to be small. By Bernoulli's equation

$$\frac{p}{\rho} = \text{constant} - gy - \tfrac{1}{2}\left\{\left(c + \frac{\partial\psi_1}{\partial y}\right)^2 + \left(\frac{\partial\psi_1}{\partial x}\right)^2\right\}$$

$$= \text{constant} - gy - c\frac{\partial\psi_1}{\partial y}, \quad \ldots\ldots\ldots\ldots(109)$$

if we neglect small terms of the second order. We assume the motion to have been originated somehow by the operation of ordinary forces, and therefore to be irrotational, so that

$$\frac{\partial^2\psi_1}{\partial x^2} + \frac{\partial^2\psi_1}{\partial y^2} = 0. \quad \ldots\ldots\ldots\ldots\ldots(110)$$

We further assume, in the first instance, that the depth is very great

compared with the other linear magnitudes with which we are concerned. The simplest solution of (110) which is periodic with respect to x, and vanishes for $y = -\infty$, is

$$\psi_1 = Ce^{ky} \sin kx. \qquad \ldots\ldots\ldots\ldots\ldots\ldots (111)$$

If we take the origin O at the mean level of the surface, the condition that the wave-profile may be a stream-line is, by (108),

$$\eta = -\frac{C}{c}e^{ky} \sin kx = -\frac{C}{c} \sin kx, \qquad \ldots\ldots\ldots (112)$$

if we neglect an error of the second order in C. We have still to secure that this is a line of constant pressure. Substituting in (109), the result will be independent of x, provided

$$g\frac{C}{c} - kcC = 0, \quad \text{or} \quad c^2 = \frac{g}{k}, \qquad \ldots\ldots\ldots\ldots (113)$$

to our order of approximation. The wave-length, i.e. the distance between successive crests or hollows is $\lambda = 2\pi/k$, so that

$$c = \sqrt{\left(\frac{g\lambda}{2\pi}\right)}. \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (114)$$

This gives the wave-velocity relative to still water.

The original form of the problem is restored if we omit the first term in (108), and replace x by $x - ct$. Thus, if we denote the surface amplitude C/c by a we have

$$\left.\begin{aligned} \eta &= a \sin k(ct - x), \\ \psi &= -ace^{ky} \sin k(ct - x). \end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots (115)$$

To find the motion of the individual particles, we may with consistent approximation write

$$\left.\begin{aligned} \frac{Dx}{Dt} &= u = -\frac{\partial\psi}{\partial y} = kac\, e^{ky_0} \sin k(ct - x)_0, \\ \frac{Dy}{Dt} &= v = \frac{\partial\psi}{\partial x} = kac\, e^{ky_0} \cos k(ct - x_0), \end{aligned}\right\} \qquad \ldots (116)$$

where (x_0, y_0) is the mean position of the particle referred to. Integrating with respect to t, and recalling (113), we have

$$\left.\begin{aligned} x &= x_0 - a\, e^{ky_0} \cos k(ct - x), \\ y &= y_0 + a\, e^{ky_0} \sin k(ct - x). \end{aligned}\right\} \qquad \ldots\ldots\ldots (117)$$

The particles therefore describe circles whose radius $a\,e^{ky_0}$ diminishes from the surface downwards. At a depth of a wave-length, $y_0 = -\lambda$, $e^{ky_0} = e^{-2\pi} = 0{\cdot}00187$. The preceding investigation is therefore practically valid for depths of the order of λ, or even less.

For smaller depths, provided they are uniform, the solution (111) is to be replaced by

$$\psi_1 = \mathrm{C}\,\sinh k(y+h)\,\sin kx, \quad\ldots\ldots\ldots\ldots(118)$$

since this makes $v = 0$ for $y = -h$. We should now find

$$c^2 = \frac{g}{k}\tanh kh = \frac{g\lambda}{2\pi}\tanh\frac{2\pi h}{\lambda}. \quad\ldots\ldots\ldots(119)$$

For small values of h/λ this gives $c = \sqrt{(gh)}$, and so verifies the former theory of long waves. As h/λ increases, $\tanh kh$ tends to unity as a limit, and we reproduce the result (114). The paths of the individual particles are ellipses whose semi-axes

$$a\frac{\cosh k(y_0+h)}{\sinh kh},\quad a\frac{\sinh k(y_0+h)}{\sinh kh}$$

are horizontal and vertical, respectively.

The energy, per unit area of the surface, of deep-water waves is found as follows. The potential energy is

$$\tfrac{1}{2}g\rho\eta^2 = \tfrac{1}{2}g\rho a^2\sin^2 k(ct-x), \quad\ldots\ldots\ldots(120)$$

the mean value of which is $\frac{1}{4}g\rho a^2$. The kinetic energy is

$$\int_{-\infty}^{0}\tfrac{1}{2}\rho(u^2+v^2)dy = \tfrac{1}{2}\rho k^2a^2c^2\int_{-\infty}^{0}e^{2ky}dy = \tfrac{1}{4}\rho ka^2c^2 \quad ..(121)$$

by (115). Since $c^2 = g/k$ the energy is, on the whole, half-potential and half-kinetic. The total energy per wave-length $(2\pi/k)$ is $\pi\rho a^2c^2$. This is equal to the work which would be required to raise a stratum of the fluid, of thickness a, through a height $\frac{1}{2}a$.

The theory of waves on the common boundary of two superposed liquids, both of great depth, is treated in a similar manner. The formulæ (108), (109), (111) may be retained as applicable to the lower fluid. For the upper fluid (of density ρ') we write

$$\psi = cy + \psi_1', \quad\ldots\ldots\ldots\ldots\ldots(122)$$

and

$$\psi_1' = \mathrm{C}'e^{-ky}\sin kx \quad\ldots\ldots\ldots\ldots(123)$$

since ψ_1' must vanish when y is very great. This makes

$$\eta = -\frac{C'}{c}\sin kx, \qquad (124)$$

We have also

$$\frac{p}{\rho'} = \text{constant} - c\frac{\partial\psi_1'}{\partial y}. \qquad (125)$$

The two values of p will be equal provided

$$g\rho\frac{C}{c} - kc\rho C = g\rho\frac{C'}{c} + kc\rho' C'. \qquad (126)$$

By comparison of (112) and (124) we have $C = C'$, and therefore

$$c^2 = \frac{g}{k}\frac{\rho - \rho'}{\rho + \rho'}. \qquad (127)$$

If $(\rho - \rho')/(\rho + \rho')$ is small, as in the case of oil over water, the oscillations are comparatively slow, owing to the relative smallness of the potential energy involved in a given deformation of the common surface. A remarkable case in point is where there is a stratum of fresh water over salt, as in some of the Norwegian fiords, where an exceptional wave-resistance due to this cause is sometimes experienced.

The preceding theory of surface-waves is restricted to the case of a simple-harmonic profile. It is true that any other form can be resolved into simple-harmonic constituents of different wave-lengths, and that it is legitimate, so far as our approximation extends, to superpose the results. But the formula (114) shows that each constituent will travel with its own velocity, so that the form of the profile continually changes as it advances. The only exception is when the wave-lengths which are present with sensible amplitude are all large compared with the depth, in which case there is a common wave-velocity $\sqrt{(gh)}$ as found above.

3. *Group Velocity*

One consequence of the dependence of wave-velocity on wave-length is that a group of waves of approximately simple-harmonic type often appears to advance with a velocity less than that of the individual waves. The simplest illustration is furnished by the

combination of two simple-harmonic trains of equal amplitude but slightly different wave-lengths, thus

$$\eta = a \cos k(x - ct) + a \cos k'(x - c't)$$

$$= 2a \cos\left(\frac{k - k'}{2}x - \frac{kc - k'c'}{2}t\right)\cos\left(\frac{k + k'}{2}x - \frac{kc + k'c'}{2}t\right) \quad \text{..(128)}$$

If k and k' are nearly equal, the first trigonometrical factor oscillates very slowly between $+1$ and -1 as x is varied, whilst the second factor represents waves travelling with velocity $(kc + k'c')/(k + k')$. The surface has therefore the appearance of a series of groups of waves separated by bands of nearly smooth water. It is evident then that the motion of each group will be practically independent of the rest. The centre of one of the groups is determined by

$$\frac{k - k'}{2}x - \frac{kc - k'c'}{2}t = 0;$$

the group as a whole is therefore propagated with the velocity

$$U = \frac{kc - k'c'}{k - k'} = \frac{d(kc)}{dk}, \quad \text{............(129)}$$

in the limit. This is called the *group-velocity*. If c is constant, as when the wave-length is large compared with the depth, we have $U = c$. On the other hand, for waves on deep water, $c^2 = g/k$, by (113), so that

$$\frac{2}{c}\frac{dc}{dk} = -\frac{1}{k},$$

whence

$$U = \tfrac{1}{2}c, \quad \text{..................(130)}$$

or the group-velocity is only one-half the wave-velocity. The general formula, obtained from (119), is

$$\frac{U}{c} = \tfrac{1}{2} + \frac{kh}{\sinh 2kh}. \quad \text{..............(131)}$$

This expression diminishes from 1 to $\frac{1}{2}$ as kh increases from 0 to ∞.

The group-velocity U determines the rate of propagation of energy across a vertical plane. To take the case of deep-water waves as simplest, the rate at which work is done on the fluid to the right

of a plane through the origin perpendicular to the axis of x is

$$\int_{-\infty}^{0} pudy. \qquad \ldots\ldots\ldots\ldots\ldots\ldots (132)$$

The value of p is given by Bernoulli's equation provided we put $q^2 = (-c+u)^2 + v^2 = c^2 - 2cu$, to our order of approximation. The only term in the resulting value of p which varies with the time is ρcu. Now

$$\rho c\int_{-\infty}^{0} u^2 dy = \rho k^2 a^2 c^3 \sin^2 kct \int_{-\infty}^{0} e^{2ky} dy = \tfrac{1}{2}\rho k a^2 c^3 \sin^2 kct \ldots (133)$$

The work done in a complete period $(2\pi/kc)$ is therefore $\frac{1}{2}\pi\rho a^2 c^2$, which is *half* the energy of the waves which pass the above plane in the same time. The apparent paradox disappears if we remember that the conception of an infinitely extended train is an artificial one. In the case of a finite train, generated by some periodic action at the origin which has only been in operation for a finite time, the profile will cease to be approximately uniform in character and sinusoidal near the front; there will be a gradual diminution of amplitude, and increase of wave-length, by which the transition to smooth water is effected. We infer from the preceding argument that the approximately simple-harmonic portion of the train is lengthened only by half a wave-length in each period of the originating force.

The principle that U rather than c determines the rate of propagation of energy holds also, not only in the case of waves on water of finite depth, but in all cases of wave-motion in Physics.

Some further results of theory must be merely stated in general terms. A localized disturbance travelling over still water with velocity c leaves behind it a train of waves whose length $(2\pi/k)$ is related to c by the formula (113) or (119), as the case may be. In the same way a stationary disturbance in a stream produces a train of waves on the down-stream side. In the former case the energy spent in producing the train measures the *wave-resistance* experienced by the disturbing agency. If E be the mean energy per unit length of the wave-train, the space in front of the disturbance gains in unit time the energy cE, whilst the energy transmitted is UE, where U is the group-velocity. The wave-resistance R is therefore given by

$$Rc = (c - U)E \ldots\ldots\ldots\ldots\ldots\ldots (134)$$

The value of E has been found to be $\frac{1}{2}g\rho a^2$, but unfortunately the value of a can be predicted only in a few rather artificial cases.

A curious point arises in the case of finite depths. It appears from (119) that the wave-velocity cannot exceed $\sqrt{(gh)}$. The above statements do not apply, therefore, if the speed of the travelling disturbance exceeds this limit. The effect is then purely local, and $R = 0$. A considerable diminution in resistance was in fact observed by Scott Russell when the speed of a canal boat was increased in this way; and an analogous phenomenon has been noticed in the case of torpedo boats moving in shallow water.

Viscosity

1. *General Equations*

The subject of viscosity is treated in Chapter III, which deals mainly with cases of steady motion where this influence is predominant. The general equations of motion of a viscous fluid have the forms

$$\rho\frac{Du}{Dt} = -\frac{\partial p}{\partial x} - \rho\frac{\partial V}{\partial x} + \mu\nabla^2 u, \quad \ldots\ldots\ldots (135)$$

with two similar equations in (v, y) and (w, z), where

$$\nabla^2 \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2.$$

The formal proof must be passed over, but an interpretation of the equations, which differ only from (88) by the terms at the ends, can be given as follows. Considering any function of the position of a point, let F be its value at P, whose co-ordinates are (x, y, z). Its value at an adjacent point $(x + \alpha, y + \beta, z + \gamma)$ will exceed its value at P by the amount

$$\frac{\partial F}{\partial x}\alpha + \frac{\partial F}{\partial y}\beta + \frac{\partial F}{\partial z}\gamma$$
$$+ \frac{1}{2}\left(\frac{\partial^2 F}{\partial x^2}\alpha^2 + \frac{\partial^2 F}{\partial y^2}\beta^2 + \frac{\partial^2 F}{\partial z^2}\gamma^2 + 2\frac{\partial^2 F}{\partial y \partial z}\beta\gamma + 2\frac{\partial^2 F}{\partial z \partial x}\gamma\alpha + 2\frac{\partial^2 F}{\partial x \partial y}\alpha\beta\right),$$

approximately. If we integrate this over the volume of a sphere of small radius r having P as centre, the first three terms give a zero result owing to the cancelling of positive and negative values of α, β, γ. The terms containing $\beta\gamma$, $\gamma\alpha$, $\alpha\beta$, also disappear for a similar reason. The mean value of α^2 or β^2 or γ^2 on the other hand

is $\frac{1}{5}r^2$, by the theory of moments of inertia. The mean value over the sphere of the aforesaid excess is therefore $\frac{1}{10}r^2\nabla^2F$. The reason why this should vary with the radius of the sphere is obvious. It is also clear that the expression ∇^2F gives a measure of the degree to which the value of the function F in the immediate neighbourhood of P deviates from its value at P. In particular ∇^2u measures the extent to which the x-component of the velocity in the neighbourhood of P exceeds the component at P. The first of the equations (135) accordingly asserts that in addition to the forces previously investigated there is a force proportional to this measure. An excess of velocity about P contributes a force tending to drag the matter at P in the direction of this excess.

The coefficient μ in (135) is called the *coefficient of viscosity*. In cases of varying motion we are often concerned not so much by the viscosity itself as by the ratio which it bears to the inertia of the fluid. It is then convenient to introduce a symbol (ν) for the ratio μ/ρ. This is called the *kinematic viscosity*.

An important conclusion bearing on the comparison of model- and full-scale experiments can be drawn from the mere *form* of these equations. Omitting the term representing extraneous force, the first equation is in full

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\nabla^2u. \quad \ldots(136)$$

Now consider another state of motion which is exactly similar except for the altered scales of space and time. Distinguishing this by accented letters, a comparison of corresponding terms in the respective equations shows that we must have

$$\frac{u'}{t'}:\frac{u}{t}=\frac{u'^2}{x'}:\frac{u^2}{x}=\frac{p'}{\rho'x'}:\frac{p}{\rho x}=\frac{\nu'u'}{x'^2}:\frac{\nu u}{x^2}. \quad \ldots(137)$$

The equality of the first two ratios requires that

$$u':u=\frac{x'}{t'}:\frac{x}{t},$$

as was evident beforehand. The equality of the second and fourth ratios requires

$$\frac{\nu'}{u'x'}=\frac{\nu}{ux}. \quad \ldots\ldots\ldots\ldots(138)$$

A necessary condition for the similarity of the two motions is therefore that Vl/ν should have the same value in both, where V is any

characteristic velocity, and l any linear dimension involved. The ratio of corresponding stresses is then

$$\frac{p'}{p} = \frac{\rho' u'^2}{\rho u^2}. \qquad\qquad (139)$$

It is to be noted that the viscous terms disappear from the equations (135) if the motion is irrotational, since we then have $\nabla^2\phi = 0$, and therefore $\nabla^2 u = 0$, $\nabla^2 v = 0$, $\nabla^2 w = 0$. But it is in general impossible to reconcile the existence of irrotational motion with the condition of no slipping at the boundary, which is well established experimentally. The above remark suggests, however, that, when the motion is started, vorticity originates at the boundary and is only gradually diffused into the interior of the fluid.

2. *Two-dimensional Cases*

The diffusion of vorticity is most easily followed in the two-dimensional case. The equations may be written, in virtue of (12), in the forms

$$\left.\begin{aligned} \frac{\partial u}{\partial t} - v\zeta &= -\frac{\partial \chi}{\partial x} + \nu^2\nabla_1 u, \\ \frac{\partial v}{\partial t} + u\zeta &= -\frac{\partial \chi}{\partial y} + \nu\nabla_1^2 v, \end{aligned}\right\} \qquad\qquad (140)$$

where
$$\chi = \frac{p}{\rho} + \tfrac{1}{2}(u^2 + v^2) + V, \qquad\qquad (141)$$

and
$$\nabla_1^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \qquad\qquad (142)$$

Differentiating the second of equation (140) with respect to x, and the first with respect to y, and subtracting and making use of the equation of continuity (7), we have, finally

$$\frac{D\zeta}{Dt} = \nu\nabla_1^2\zeta. \qquad\qquad (143)$$

This is exactly the equation of conduction of heat, with the vorticity ζ in place of the temperature, and the *kinematic viscosity* $\nu(= \mu/\rho)$ in place of the *thermometric conductivity*. Consequently, various known results in the theory of conduction can be at once utilized in the present connection.

For instance, the known solution for the diffusion of heat from an initially heated straight wire into a surrounding medium can be applied to trace the gradual decay of a line vortex initially concentrated in the axis of z. Since there is symmetry about Oz the equation (143) takes the form

$$\frac{\partial \zeta}{\partial t} = \nu\left(\frac{\partial^2 \zeta}{\partial r^2} + \frac{1}{r}\frac{\partial \zeta}{\partial r}\right), \quad \ldots\ldots\ldots\ldots (144)$$

as may be seen by a comparison of the left-hand members of (25) and (26). It is easily verified by differentiation that this equation is satisfied by

$$\zeta = \frac{\kappa}{4\pi\nu t}e^{-r^2/4\nu t}, \quad \ldots\ldots\ldots\ldots\ldots (145)$$

which vanishes for $t = 0$ except at the origin. Moreover, this gives for the circulation in a circle of radius r

$$\int_0^r \zeta . 2\pi r dr = \kappa(1 - e^{-r^2/4\nu t}). \quad \ldots\ldots\ldots (146)$$

As t increases from 0 to ∞, this sinks from κ to 0. The value of ζ, on the other hand, at any given distance r increases from zero to a maximum and then falls asymptotically to zero.

A comparatively simple application of the equations of motion is to the case of "laminar" flow in parallel planes, or of smooth rectilinear flow in pipes, but the results have only a restricted application to actual phenomena. To take an example due to Helmholtz, consider the flow of a hypothetical atmosphere of uniform density, and height H, over a horizontal plane. If it is subject to inertia and viscosity alone, the equation of motion is

$$\frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial y^2}, \quad \ldots\ldots\ldots\ldots\ldots\ldots (147)$$

with the conditions that $u = 0$ for $y = 0$ and $\partial u/\partial y = 0$ for $y = \mathrm{H}$. These are all satisfied by

$$u = \mathrm{A}e^{-\nu k^2 t}\sin ky, \quad \ldots\ldots\ldots\ldots\ldots (148)$$

provided

$$\cos k\mathrm{H} = 0, \quad \text{or} \quad k = (2n+1)\frac{\pi}{2\mathrm{H}}, \quad \ldots\ldots (149)$$

where n is an integer. By addition of such solutions with different values of n and suitable values of the coefficients A we can represent

the effect of any initial state, e.g. one of uniform velocity. The most persistent constituent in the result is that for which $n = 0$. This will have fallen to one-half its original value when

$$\nu k^2 t = \log 2, \quad \text{or} \quad t = \frac{4 \log 2}{\pi^2} \frac{H^2}{\nu} \text{.........} (150)$$

Putting $\nu = 0{\cdot}134$ (air), $H = 8026$ metres, this makes $t = 305{,}000$ years! The fact is that in such a case the laminar motion would be unstable; turbulent motion would ensue, by which fresh masses of fluid moving with considerable velocity are continually brought into contact with the boundary, so that the influence of viscosity is enormously increased.

CHAPTER III

Viscosity and Lubrication

A. VISCOSITY

All motions of actual fluids, as distinguished from the " perfect fluid " of the mathematician, are accompanied by internal forces which resist the relative movements and are therefore analogous to frictional forces between solid bodies. The origin of the frictional resistances is in all cases referred to the property of *viscosity*, common in varying degree to all fluids, which has already been defined in general terms in Chapter I. The present chapter is devoted to a fuller explanation of the theory of this property and to discussions of some of its direct applications, one of the chief of these being to the theory of lubrication.

There are other direct applications of the theory of viscosity which are of importance to engineers, most though not all of which relate to the motion of fluids in narrow channels or in thin layers between solid surfaces, and these applications are met with in all branches of engineering. The fluid frictions, however, which chiefly concern hydraulic and other engineers, who deal with fluids such as water or air in large volumes, though physically referable in origin to viscosity, cannot be directly calculated by means of its theory. The appropriate methods applicable to such cases are discussed in Chapters IV and V. In the meantime it may be said of the *direct* applications of the theory, in Rayleigh's words (20, p. 159),* that in these cases " we may anticipate that our calculations will correspond pretty closely to what actually happens—more than can be said of some branches of hydrodynamics ".

* Arabic numerals in brackets after names of authors refer to the short bibliography at the end of this chapter.

Laminar Motion

The law of viscous resistance is most clearly conceived in the case of *laminar motion*, which may be defined as a state of motion of a body of fluid in which the direction of the motion of the particles is the same at all points and the velocity is the same throughout each of a series of planes parallel to one another and to the direction of motion. A volume of fluid in laminar motion can thus be roughly regarded as a series of very thin layers of solid material, sliding one upon another in a common direction. Quantitatively, if the face δx, δy of the rectangular element δx, δy, δz (fig. 1) is parallel to the laminæ, and if the laminar motion is in the direction of X, the velocity of flow, u, at any point P, will depend only on the distance, z, of the point P from the plane XY. If the element is sufficiently small, u may be taken as varying uniformly with z over the small distance δz, so that if u_0, u_1 are respectively the velocities of the laminæ which form the lower and upper faces of the element $u_1 = u_0 + \frac{\partial u}{\partial z}\delta z$, in which $\frac{\partial u}{\partial z}$ can be regarded as constant over the small distance δz.

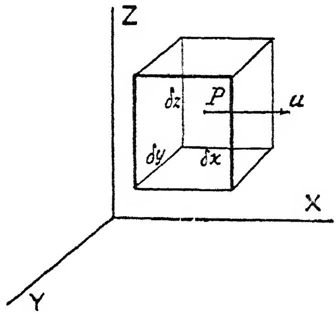


Fig. 1

In a viscous fluid there will then be exerted a shearing force, or traction, parallel to X, between the portions of the element of fluid above and below any section of the element parallel to the face $\delta x \delta y$, tending to retard the portion which is moving with the higher velocity, and the magnitude of this force will be

$$S = \mu\frac{\partial u}{\partial z}\delta x\delta y, \qquad \text{(1)}$$

μ being a quantity, independent of x, y, z, u, and $\frac{\partial u}{\partial z}$, known as the *coefficient of viscosity*.

Coefficient of Viscosity

The value of the quantity μ varies greatly from one fluid to another, and in any one fluid it changes with the temperature, and to a smaller extent with the pressure, of the fluid. Its value is in general much higher for liquids than for gases. Liquids in which the value of μ is low are said to be " limpid ", " thin ", or " light ", while those in which it is comparatively great are said to be " viscous ", " thick " or " heavy ". There is however no necessary, or general, correspondence between the density of a liquid and its viscosity. Thus mercury, the heaviest of known liquids at atmospheric temperatures, is one of the least viscous.

The fact that the coefficient of viscosity, for a given liquid at constant temperature, is independent of the rate of shear was first experimentally proved with great accuracy by Poiseuille (1), not, however, by direct measurement of plane laminar flow, but by investigation of the flow of water in small cylindrical tubes.

The flow of fluids in such tubes, as well as the motion of viscous fluids in many other cases which are of practical interest, is closely analogous to plane laminar flow.

The importance of the coefficient of viscosity μ, however, arises from the fact that it is the sole physical constant connecting the internal frictional resistances of fluids with their relative motions, not only in the case of such simple types of motion, but of all kinds of fluid motion however complicated, provided that they are not discontinuous or unstable.

The explanation of this unique property of the coefficient of viscosity requires some analysis of the types of deformation of which a fluid element is susceptible. This analysis is given briefly in the following paragraphs, from which it will be seen that the relations between the internal motions and stresses in a fluid are similar to, but essentially simpler than, those between the deformations and stresses in an elastic body. The failure, already referred to, of the law of viscosity when fluid motions become discontinuous or unstable may be regarded as analogous to the failure of the laws of elasticity in solids when fracture takes place or the " yield-point " is exceeded. In such cases the conditions which result are no longer amenable to theoretical calculation.

We proceed to show that, when no such discontinuities exist, there is in fluids only one kind of internal resistance and only one coefficient of viscosity.

Relative Velocities

If u, v, w (see fig. 2) are the components of the velocity parallel to the rectangular axes X, Y, Z of a particle of the fluid at the point x, y, z, the corresponding components for the neighbouring point $x + \delta x$, $y + \delta y$, $z + \delta z$ are

$$\left.\begin{aligned} u' &= u + \frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y + \frac{\partial u}{\partial z}\delta z, \\ v' &= v + \frac{\partial v}{\partial x}\delta x + \frac{\partial v}{\partial y}\delta y + \frac{\partial v}{\partial z}\delta z, \\ w' &= w + \frac{\partial w}{\partial x}\delta x + \frac{\partial w}{\partial y}\delta y + \frac{\partial w}{\partial z}\delta z; \end{aligned}\right\} \quad (2)$$

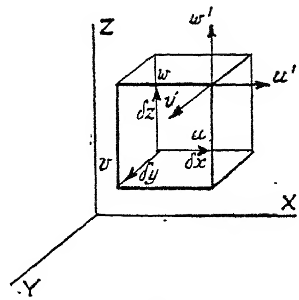


Fig. 2

and the components of the velocity of the second point relatively to the first are $u' - u$, $v' - v$, $w' - w$, or

$$\left.\begin{aligned} &\frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y + \frac{\partial u}{\partial z}\delta z, \\ &\frac{\partial v}{\partial x}\delta x + \frac{\partial v}{\partial y}\delta y + \frac{\partial v}{\partial z}\delta z, \\ &\frac{\partial w}{\partial x}\delta x + \frac{\partial w}{\partial y}\delta y + \frac{\partial w}{\partial z}\delta z. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (3)$$

Of the derivatives in these expressions it is clear from inspection of fig. 2 that $\frac{\partial u}{\partial x}$, $\frac{\partial v}{\partial y}$, and $\frac{\partial w}{\partial z}$ represent rates of stretching or elongation of the element in the directions of X, Y, and Z respectively, while by the pairs of sums of derivatives:

$$\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}, \quad \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}, \quad \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y},$$

are represented respectively rates of change of the angles between the edges δy and δz, δz and δx, and δx and δy of the element.

Thus by means of these six expressions any deformation of the element can be expressed.

As a hypothesis which is suggested as probable by the experimental law proved by Poiseuille, but which depends for its real justifi cation on the consistent correspondence of the results of theory with experience, it is assumed that the frictional forces arise from

the rates of deformation of the elements of the fluid, and are linear functions of these rates.

As to the three rates of elongation $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial w}{\partial z}$, it is a well-known theorem that they can be resolved into a rate of dilatation or compression of the elementary volume, uniform in all three directions, combined with three rates of shearing deformation respectively in the directions of the diagonals of the faces of the element supposed cubical.*

As there is no experimental evidence of any internal resistances, either in liquids or gases, depending on rates of change of volume by dilatations or compressions equal in all directions, resistance to an elongation, such as $\frac{\partial u}{\partial x}\delta x$, can only arise from its shearing components. Such resistances are therefore of the same kind as those which depend on the purely shearing deformations whose rates are $\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}$, &c.

In the applications which follow, the axes X, Y, Z will be so chosen that the rates of elongation, such as $\frac{\partial u}{\partial x}$, and consequently also their component rates of shear, are everywhere small compared to the rates of shear represented by $\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}$, &c.

Now in a homogeneous liquid or gas there is no physical difference in the properties depending on the direction of the co-ordinates, consequently (the frictional forces being linear functions of the rates of shear) the only forces that will arise may be expressed as:

$$\left.\begin{aligned} \dagger\, S_{yz} &= \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right), \\ S_{zx} &= \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right), \\ S_{xy} &= \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right), \end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots \quad (4)$$

involving the single coefficient μ. By comparison with fig. 1, and

* Cf. the similar theorem for stresses (Morley, *Strength of Materials*, 2nd ed., p. 12).

† In this notation the first subscript indicates the direction of the normal to the plane on which the force acts; the second the direction in which the force acts—thus S_{yz} is the shearing stress on a plane a normal of which is parallel to Oy and S_{yz} acts in the Oz direction.

of equation (1) with the second of the above equations (4), in which $\frac{\partial w}{\partial x}$ is taken as zero, it is seen that this constant is the same as the coefficient μ introduced in the special case of laminar motion.

Conditions at the Bounding Surfaces of Fluids

Before the laws of fluid friction can be applied to fluids as we actually have to deal with them, account must be taken of the behaviour of the fluid where it is in contact with the solid bodies which contain it. In the case of liquids the condition of a free upper surface, usually a surface of contact with air at atmospheric pressure, has also to be considered.

It is clear in the first place that the presence of a boundary involves that on the bounding surface the relative velocity of the fluid normal to that surface is zero. The normal velocity will furthermore be very small at all points *near* the bounding surface. For, let W, fig. 3, be the fixed bounding surface, and W′ a surface in the fluid parallel, and very near, to W. For simplicity W and W′ may be considered plane. Let the average velocity towards W over a circle of radius R in the plane W′ be v, the normal distance between W and W′ being δn. Then a volume of fluid $\pi R^2 v$ flows through this circular area in unit time. In the same time a volume $2\pi R \delta n . \sigma$ flows outward between the surfaces past the circumference of the circle, σ being the mean outward radial velocity parallel to the surfaces. Thus

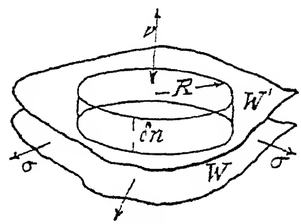


Fig. 3

$$v = 2\sigma\frac{\delta n}{R}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (5)$$

or the normal velocity is very small compared to the velocity parallel to the surface.

In the case of a solid boundary it will be seen from the next paragraph that the velocity σ is itself very small close to the surface, so that in this case the normal velocity v is a small quantity of the *second* order.

Motion Parallel to Bounding Surfaces

With regard to the motion of fluids parallel to solid walls with which they are in contact, there is strong evidence that in the case of liquids at least the relative tangential velocity σ at the wall is zero. Some of the evidence will be referred to later in connection with the flow of liquid through tubes under great pressure, and in the discussion of the theory of lubrication.

Even when the mutual molecular attraction of a liquid and solid appears to be comparatively small, so that the liquid does not tend to spread over, or " wet " the surface of the solid, as is the case with mercury and glass, there is no observable sliding or slipping of the fluid over the solid at their common surface.

If the tangential tractional force between liquid and solid, and consequently the rate of shear in the liquid near the surface, are finite, the relative tangential velocity, being zero at the surface, must be still small at all points of the liquid near the surface, as was asserted in the last paragraph.

In gases, the same rule as to the relative velocity being zero at a solid surface is found to apply under ordinary circumstances, at least as a very close approximation. When, however, a gas is at such low pressure that its molecules are at distances apart comparable with the dimensions of the volume of gas which is being dealt with, phenomena are observed which can be regarded as arising from an appreciable velocity of slipping of the gas over the solid surface. According to Maxwell,* the motion of the gas is very nearly the same as if a stratum, of depth equal to twice the mean free path of the gas molecules, had been removed from the solid and filled with the gas, there being no slipping between the gas and the new solid surface.

At free surfaces, which, of course, can only exist in liquids, the normal velocity relative to the surface is again obviously zero. The liquid surface may, however, have a tangential velocity, and it is usual to assume that the law of viscous shear holds up to the surface and that either the tangential traction there becomes zero, or, if the liquid surface is exposed to a stream of air, that the traction is due only to the rate of shear in the air near the common surface. Experiments by Rayleigh† and others have shown that, at least in the cases of water with an uncontaminated surface and of oils and other liquids which are capable of dissolving solid grease films, there are no frictional resistances peculiar to the surface film.

* *Collected Papers*, Vol. II, p. 708. † *Collected Papers*, Vol. III, p. 363.

Viscous Flow in Tubes

On the principles which have been explained, we can proceed to calculate the flow of viscous fluids in various cases which are of practical interest. Take first the case of a uniform tube of circular section of which the diameter is small compared to the length of the tube. A fluid flows through the tube as the result of a constant difference of pressure between its two ends. The motion, except very near the ends, will be sensibly parallel to the axis of the tube, and the pressure (and consequently the density) will be sensibly uniform over every normal section. By symmetry, at any one section the velocity must be the same at all points at any given radius r from the axis.

If w be the velocity (upwards in fig. 4) at this radius, ρ the density, and p the pressure at any section, the radius of the bore of the tube being a, the mass discharged per unit time, which must be the same for all sections, is

$$\int_0^a \rho w . 2\pi r dr = m, \text{ constant. .... (6)}$$

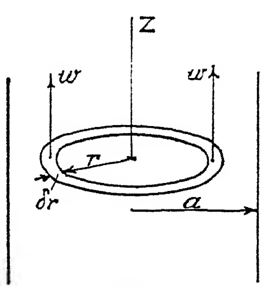


Fig. 4

The axis of the tube is taken as the axis Z, and is supposed to be so nearly straight that effects due to its curvature can be neglected, and in the first instance the motion will be supposed so slow that the kinetic energy of the fluid is inappreciable. The fluid may be either a liquid or gas. The effect of gravity is disregarded, or, if included, $p + g\rho z$ is to be written instead of p.

From equations (4), p. 106, since the velocity w varies radially at the rate $\frac{\partial w}{\partial r}$, but not circumferentially, there is a traction in the direction of Z on each unit of area of the cylindrical body of fluid inside radius r of amount

$$S_{rz} = \mu \frac{\partial w}{\partial r} \ldots\ldots\ldots\ldots\ldots\ldots \quad (7)$$

Considering a section of this cylinder of length δz, the total traction on its cylindrical surface, whose area is $2\pi r \delta z$, must be equal to the

difference of the total pressures on its upper and lower ends, so that

$$2\pi r S_{rz}\delta z = 2\pi r\mu\frac{\partial w}{\partial r}\delta z = \pi r^2\frac{dp}{dz}\delta z,$$

$$\text{or } \frac{\partial w}{\partial r} = \frac{r}{2\mu}\frac{dp}{dz},$$

$$\text{and therefore } w = \frac{1}{2\mu}\frac{dp}{dz}\left(\frac{r^2}{2} - C\right).$$

Since $w = 0$ when $r = a$, $C = \dfrac{a^2}{2}$, and

$$w = -\frac{1}{4\mu}\frac{dp}{dz}(a^2 - r^2). \quad \ldots\ldots\ldots\ldots \quad (8)$$

in which the negative sign expresses the obvious fact that the direction of flow is opposite to the direction of increase of pressure.

Now from (6) and (8)

$$m = \int_0^a \rho w.2\pi r dr = -\int_0^a \frac{\rho}{4\mu}.2\pi r(a^2 - r^2)\frac{dp}{dz}dr$$

$$= -\frac{2\pi\rho}{4\mu}\left[\frac{a^2r^2}{2} - \frac{r^4}{4}\right]_0^a\frac{dp}{dz}$$

$$= -\frac{\pi\rho a^4}{8\mu}\frac{dp}{dz}.^* \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (9)$$

In the case of a liquid, ρ and μ may usually be taken as constant, so that $\dfrac{dp}{dz}$ is constant along the length of the tube, being equal to $\dfrac{p_2 - p_1}{l}$ where p_1, p_2 are the pressures at the lower and upper ends of the tube whose length is l.

$$\text{Then } m = \frac{\pi a^4\rho}{8\mu}\frac{p_1 - p_2}{l} \quad \ldots\ldots\ldots\ldots \quad (10)$$

The limits of application of this formula will be more fully explained in a later chapter. For the present it may be stated to be applicable to the flow of all liquids through "capillary" tubes (that is to say, tubes whose diameter is only a fraction of a millimetre), unless the

* In all numerical applications of this and other formulæ throughout this chapter all quantities must be expressed in the C.G.S. or other absolute system of units.

difference of pressures, $p_1 - p_2$, is greater than is ordinarily met with in engineering practice, provided that proper correction is made for the disturbing effects of the ends of the tube.

In the case of viscous lubricating oils, the formula is applicable, with certain restrictions, to their flow through ordinary lines of piping, but it must be regarded as subject to correction, or even wholly inapplicable, to the flow of the less viscous oils especially under considerable pressures.*

In the case of a gas, $\rho = \frac{p}{RT}$, T being the absolute temperature and R a constant. Thus from (9)

$$mdz = -\frac{\pi a^4}{8\mu RT}pdp \qquad (11)$$

If T and μ can be regarded as constant throughout the length of the tube, integrating (11) we have

$$ml = \frac{\pi a^4}{16\mu RT}(p_1^2 - p_2^2) \qquad (12)$$

as the equation connecting the flow and the fall of pressure.

In the preceding discussion the kinetic energy of the fluid has been assumed to be negligible. All the formulæ given, however, remain correct for the case of a liquid even when the kinetic energy is appreciable, provided that they are applied only to the middle portion of the tube and not to its end portions where the flow is affected by the acceleration and retardation of the fluid which occur near the inlet and outlet. It is well known that the kinetic energy which a fluid acquires in entering an orifice is not wholly restored as pressure energy at its discharge. There is therefore a resistance to the flow arising from the acceleration and retardation at the inlet and outlet of a tube, additional to the frictional losses within the tube itself. In the case of a square-ended tube opening into large vessels at each end, the loss of pressure is approximately $1{\cdot}12 \times \bar{u}^2/2g$, where $\bar{u}$ is the mean velocity at the outlet.†

There are further sources of resistance not taken into account in our calculations, arising from viscous friction between the streams at the ends, where the lines of flow are not parallel to the axis of the tube. Fig. 5‡ shows the course of the particles of fluid at the inlet and outlet of a square-ended tube when the kinetic energy is appreciable and both ends of the tube are immersed in the fluid.

* See e.g. (13), p. 159.

† See Hosking, *Phil. Mag.*, April, 1909; Schiller, *Zeits. Math. u. Mech.*; Bond, *Proc. Phys. Soc.*, 34, IV.

‡ From (10), p. 158.

Fig. 6 illustrates the condition which occurs when the outlet of such a tube is not immersed but discharges the fluid in a series of drops. In this case there is another resistance to the flow, due to the excess of internal pressure which is necessary to extend the surfaces of the drops during their formation.

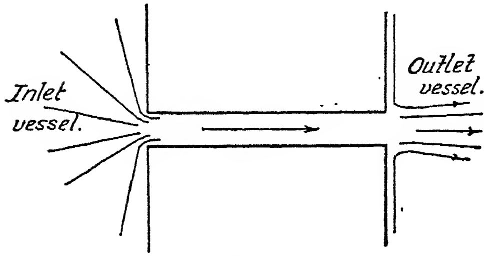


Fig. 5

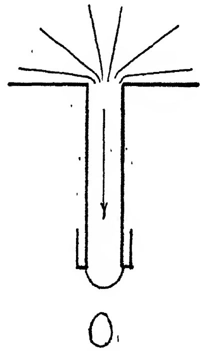


Fig. 6

The calculation of the resistances due to these disturbing effects is rather uncertain, and on this account an accurate correspondence between the results of calculation and those of experiment can only be expected when the tubes are very long compared to their diameters.

Use of Capillary Tubes as Viscometers

The experimental determination of coefficients of viscosity is carried out by instruments of various kinds, known as " viscometers " or " glischrometers ". These are divided into two classes, namely " absolute " viscometers, by the use of which the coefficient of viscosity can be determined in absolute measure directly from the dimensions of the instrument itself (combined with measurement of a time interval), and " secondary " or " commercial " viscometers, which require to be calibrated by comparison of their results with those of an " absolute " viscometer.

The best absolute viscometers, for liquids at least, depend on the measurement of flow through capillary tubes, μ being determined from the equation (10) given on p. 110, after instrumental measurement of the other quantities involved. The apparatus by which Poiseuille made the first accurate determinations of the viscosity of water was of this class. The tubes which he used varied in diameter from 0·001 to 0·014 cm., their lengths being a few centimetres, and the pressure was applied by a column of mercury up to 77 cm. in height. Such instruments are capable of very considerable accuracy

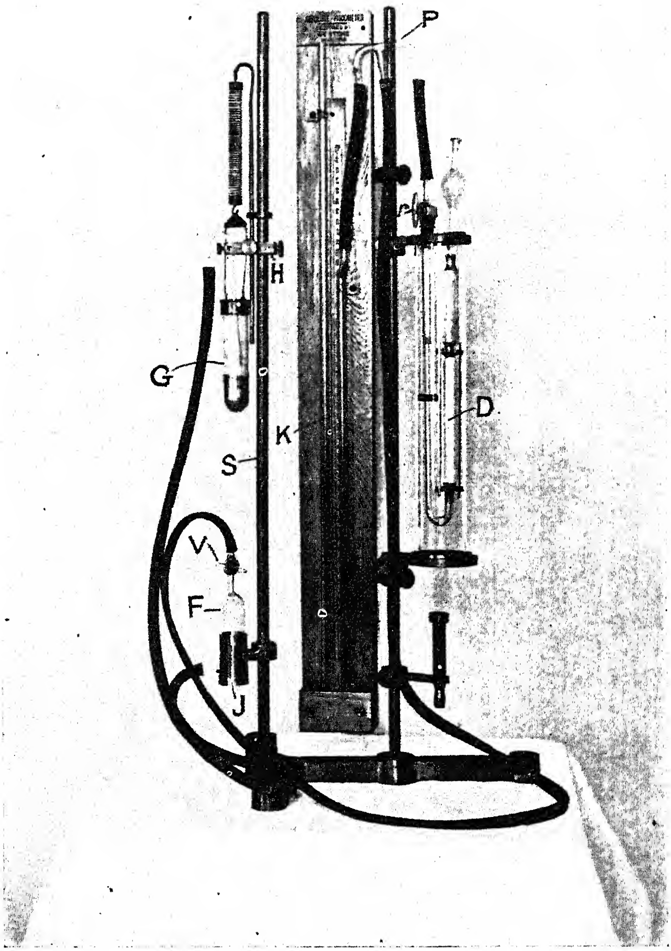


Fig. 7.—Stone's Absolute Viscometer

when used with proper precautions, and when the necessary corrections are applied for the various disturbing factors. The viscosity of water, for instance, at atmospheric temperatures is probably known within $\frac{1}{10}$th of 1 per cent of its true value.*

* See (11), p. 158.

The consistency to this order of accuracy of determinations made with different instruments and under different conditions is conclusive evidence of the correctness of the basic assumption of the linear connection of traction with shear, and of the absence of slipping of the fluid over the walls of tubes. The principal precaution which has to be taken in the use of the capillary viscometer, in addition to the elimination of (or, in so far as that is not possible, the correction for) the end-disturbances which have been pointed out, is the accurate determination of the temperature of the fluid under test. The latter requirement is usually met by surrounding the capillary tube with a water-jacket, means being provided for warming or cooling the water, and measuring its temperature.

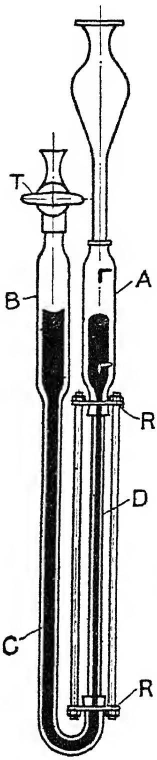


Fig. 8

The most convenient form of "absolute" capillary viscometer for liquids is that described by W. Stone (18, p. 159). In this instrument the pressure is applied by a column of mercury of which the height is automatically maintained constant, and other devices are provided which further simplify the manipulation of the instrument and the calculation of the results from the observations. The Stone viscometer is illustrated in fig. 7, the capillary tube and its attachments being shown separately in fig. 8. The following is an abbreviation of the designer's description cited above.

The instrument consists of three essential elements, viz. the viscometer burette, the adjustable constant-head apparatus, and the pressure-gauge. The viscometer burette consists of two glass vessels A and B (fig. 8), of equal internal diameters and suitable lengths, connected at their lower ends by means of a wide-bore tube C, and of a capillary tube D of suitable dimensions for the desired purpose. The three portions of the burette are held together by the brass clips and tension-rods R. Several interchangeable tubes D may be provided for fluids of different viscosities.

The measuring vessel A is provided with two platinum wires sealed into its wall, and so bent that the inner end of each wire lies on the axis of the tube. The capacity of the vessel between the two platinum points can be thus accurately measured. A glass tap T

is provided on the inlet to the burette to control the starting of a test. The whole of the burette is immersed in water contained in a glass tube (see fig. 7) having a brass bottom. A brass cover is also fitted having a slot for the insertion of a stirring rod and a thermometer. A Bunsen burner serves to heat the water.

The adjustable constant-head apparatus consists of two glass vessels, the lower one F being furnished with a tap V at the top and the upper one G suspended by a spring from a hook attached to a sliding clip H which can be clamped to the standard S at any desired height. Through the outer end of the clip a glass siphon pipe passes to the bottom of the vessel G when the latter is at its highest point, i.e. against the clip H. The siphon is connected to the lower vessel F by means of a rubber tube. The strength of the spring is so adjusted that as the mercury flows from G to F, the former, being thereby lightened, will rise so as to maintain the surface of the mercury in it at constant height above that of the mercury in F.

The pressure-gauge K is of the ordinary U pattern, with mercury as the working fluid. A three-branch pipe P connects the burette, pressure-gauge, and constant-head apparatus.

The instrument must be set up vertical. As the liquid to be tested is fed into the burette A (fig. 8), the vessel F is removed from the socket J and raised to a sufficient height above G to reduce the air-pressure in B (fig. 8) and thus draw the liquid under test into it, lowering the surface in A below the lower platinum gauge-point. The glass tap T is then closed and the pressure apparatus adjusted to the desired pressure. Then the tap is opened, and the time elapsing between the moments of contact of the liquid surface with the gauge-points in A is taken by means of a stop-watch or suitable chronograph.

By the use of this instrument the viscosity of a sample of oil can be determined at ten or twelve different temperatures within an hour. The pressure can be varied from about 5 to 50 cm. of mercury in order to give (without changing the tube D) convenient intervals of time for measurement according to variations in the viscosity of the oil.

Various other forms of apparatus have been used for the absolute determination of viscosities, their action depending, for instance, on the torsional oscillations of a disc or cylinder (a method which is convenient for measurement of the viscosity of gases, on account of the accuracy with which the very small forces involved may be

measured by this means), the continuous rotation of a cylinder or disc or sphere, or the free fall of a sphere in a body of fluid. For general purposes, however, no other method is so convenient or accurate for absolute measurements of viscosity as that of Poiseuille.

Secondary or Commercial Viscometers

Tube viscometers are also commonly employed for making practical or commercial measurements of viscosity. In order to reduce the time occupied by the measurements, and to simplify the apparatus and to reduce its delicacy, much shorter tubes are used in these instruments than are admissible for absolute instruments.

In the Redwood viscometer, for instance, the tube is approximately 1·7 mm. in diameter, and 12 mm. in length, being a hole drilled through an agate plug fixed in the bottom of a vessel which is arranged to contain a measured quantity of the liquid to be tested. The liquid flows out of the hole under the force of gravity, the time of efflux of the measured quantity being taken by a stop-watch. Means are provided for warming or cooling the liquid to any temperature at which it is desired to make the test, but the determination of the actual temperature of the fluid as it is passing through the hole is one of the chief difficulties in the use of this and similar instruments.

In some of these the " tube " is so much reduced in length as to become a mere orifice. It will be readily understood that the corrections for the end effects of the tube, which have been pointed out as necessary in connection with all cases of viscous flow in tubes, become relatively much more considerable in the case of such short-tube instruments. In these, except for the more viscous liquids, the times of efflux are no longer proportional to the viscosity of the fluid. It is therefore necessary, in order to obtain reasonably accurate results, that such instruments should be calibrated over the range of their intended application by comparison with an absolute viscometer. Such a system of calibrations not having been generally adopted, an unfortunate practice has become common of expressing viscosities, not in terms of physical or engineering units (by which alone the value of the unit can be applied in calculations), but by the number of seconds or minutes required for the efflux of a certain volume through

one or other of the best-known forms of commercial viscometers. There are thus in use as many arbitrary, irreconcilable, and dynamically meaningless units of viscosity as there are manufacturers of commercial viscometers.

A different type of secondary viscometer recently introduced is the cup-and-ball viscometer. The action of this instrument depends on the viscous flow of the fluid, not in a tube, but between two nearly parallel and closely adjacent surfaces. The instrument and its mode of operation will be more fully described below, after discussion of the theory of that type of viscous motion.

Coefficients of Viscosity of Various Fluids

In Table I (p. 118) are given values of the viscosity constant μ of a few of the fluids which are of chief interest to engineers, especially in connection with lubrication. The table contains also approximate numerical data, for the same fluids, of certain other physical properties, the significance of which, as affecting the utility of the fluids as lubricants, will be made more apparent by the later portions of this chapter. The constants are expressed in all cases in C.G.S. units. The value of μ for instance is the ratio of a stress measured in dynes per square centimetre to a rate of shear measured in centimetres per second per centimetre.

The values of μ are given for various temperatures between 0° and 100° C. The other constants, which for the most part do not vary rapidly with temperature, are stated for atmospheric temperatures in the neighbourhood of 15° C. The rule which is apparent from the table as to the values of μ for liquids, namely that the value for each liquid diminishes as the temperature rises, is true generally. It will be noticed that the rate of variation is much less rapid for mercury and carbon bisulphide than for the other liquids. In all gases, as in air, on the other hand the viscosity increases with the temperature.

Variation of Viscosity with Pressure

The viscosity of both liquids and gases varies very little with variations of pressure over a range from many times less, to many times greater, than atmospheric pressure. At pressures, however, of the order of intensity of hundreds of atmospheres most liquids appear to have greatly increased coefficients of viscosity.

TABLE I

Coefficients of Viscosity, &c., of various Fluids in C.G.S. Units (at Atmospheric Pressure)

Fluid.	Coefficient of Viscosity, μ.						Surface Tension in Air.	Density.	Specific Heat.	Thermal Conductivity.	Freezing (F) or Setting Point (S).	Boiling (B) or "Flash" Point (F).
	0° C.	20° C.	40° C.	60° C.	80° C.	100 °C.						
Water	0·0179	0·0101	0·0066	0·0048	0·0036	0·0028	75	1·00	1·00	13×10^{-4}	0°(F)	100° (B)
Mercury	0·0170	0·0157				0·0122	527	13·55	0·033	197×10^{-4}	− 39°(F)	357° (B)
Carbon bisulphide ..	0·0044	0·0038	0·0032	—	—	—	30	0·96	0·240	$3·4 \times 10^{-4}$	− 116°(F)	47° (B)
Glycerine	46	8·7					63	1·26	0·576	$6·4 \times 10^{-4}$	—	—
Sperm oil	1·13	0·335	0·173	0·079	0·040		38	0·878	0·493		0°(S)	150° (F)
Castor oil	—	7·24	2·23	0·68	0·28	0·12	38	0·969	0·508	$3·95 \times 10^{-4}$	− 15°(S)	205° (F)
Mineral Oils												
A "turbine-bearing oil"	—	1·528	0·245	0·110		0·035		0·88				
A "heavy" bearing oil ("Bayonne") ..	6·40	1·72	0·54	0·22	0·12	0·06	36	0·91	0·460	$3·5 \times 10^{-4}$	0°(S)	190° (F)
A "cylinder" oil ("Mobiloil BB") ..	—	9·47	1·30	0·54	0·26	0·12		0.91			3°(S)	185° (F)
Air	$1·73 \times 10^{-4}$	$1·86 \times 10^{-4}$	$2·01 \times 10^{-4}$	$2·1 \times 10^{-4}$	$2·2 \times 10^{-4}$	$2·3 \times 10^{-4}$	—	$1·29 \times 10^{-3}$	0·170 C_v. 0·238 C_p.	$5·68 \times 10^{-5}$	—	—

The following table (Table II) from Hyde (21, p. 159) shows how the viscosities of a few lubricating oils vary with pressures of this order. Although such pressures do not usually exist in ordinary bearings, there are cases in the application of the theory of viscosity, as will be seen later in this chapter, in which the changes of the viscous constant by increase of pressure cannot be neglected.

Within very wide limits, the viscosity of gases is independent of pressure, the viscosity of air for instance being practically invariable from a pressure of a few millimetres of mercury up to pressures of many atmospheres. This law, originally predicted by Maxwell from the kinetic theory of gases, has been confirmed by numerous experiments.

TABLE II

VISCOSITIES OF VARIOUS LUBRICATING OILS AT VARYING PRESSURES. TEMPERATURE 40° C.

Abstracted from a table by J. H. Hyde. *Proc. Roy. Soc.*, A. **97**

Pressure, Kilo-grams per Square Centimetre.	Mineral Oil ("Bayonne").	Trotter Oil (Animal).	Rape Oil (Vegetable).	Sperm Oil.
	Coefficient of Viscosity, μ, C.G.S.			
0	0·47	0·344	0·375	0·154
157·5	0·62	0·413	0·422	0·190
315·0	0·92	0·550	0·539	0·236
472·5	1·32	0·686	0·703	0·299
630·0	1·86	0·824	0·880	0·368
787·5	2·51	—	1·089	—
945	3·65	1·217	1·310	—
1102·5	5·32	—	1·578	0·619
1260	7·55	1·731	—	—

Viscous Flow between Parallel Planes

As one of the typical conditions of flow met with in problems of lubrication and other practical applications of the theory of viscosity, it is convenient to consider in detail the flow of viscous liquid between two parallel and closely adjacent plane walls, supposed fixed.

In rectangular co-ordinates, let $z = 0$, and $z = h$ be the parallel planes, h being small compared to their dimensions in the X and Y directions, as indicated in fig. 9.

For the reasons already explained the components of velocity

normal to the planes must be everywhere negligible. In other words, the rates of shear and the momentum in the Z direction are very small; consequently the fluid pressure p does not vary in that direction but is a function of x and y only. Also the rates of change from the values of the finite velocity components, u, v, in the fluid to their values, known to be zero, on the walls are rapid compared to their rates of change in the X and Y directions. Thus, considering a rectangular element, as in fig. 10, anywhere between the

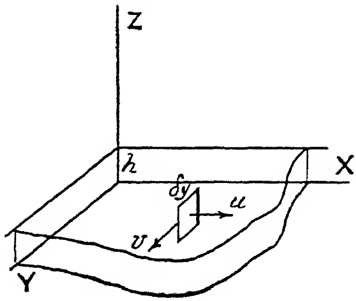


Fig 9

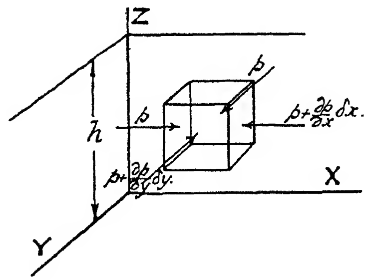


Fig. 10

planes $z = 0$ and $z = h$, the viscous tractions on its lower face in the directions in which x and y increase, are:

$$-\mu\frac{\partial u}{\partial z}\delta x\delta y,$$

$$\text{and} \quad -\mu\frac{\partial v}{\partial z}\delta x\delta y.$$

The corresponding tractions on the upper face are

$$\mu\left(\frac{\partial u}{\partial z}+\frac{\partial^2 u}{\partial z^2}\delta z\right)\delta x\delta y,$$

$$\text{and} \quad \mu\left(\frac{\partial v}{\partial z}+\frac{\partial^2 v}{\partial z^2}\delta z\right)\delta x\delta y.$$

The sums of these pairs of tractions added to the differences of the fluid pressures on the faces parallel to the YZ and ZX planes are respectively equal to the rates of increase of the momentum of the element in the X and Y directions, thus

$$\left(\mu\frac{\partial^2 u}{\partial z^2}-\frac{\partial p}{\partial x}\right)\delta x\delta y\delta z = \rho\delta x\delta y\delta z\frac{du}{dt},$$

$$\left.\begin{aligned} \text{or} \quad \mu\frac{\partial^2 u}{\partial z^2} &= \frac{\partial p}{\partial x} + \rho\frac{du}{dt}, \\ \text{and similarly} \quad \mu\frac{\partial^2 v}{\partial z^2} &= \frac{\partial p}{\partial y} + \rho\frac{dv}{dt}, \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots(13)$$

ρ being the density of the liquid.

The rates of increase of velocity $\frac{du}{dt}$, $\frac{dv}{dt}$ are of the order of the products $u\frac{\partial u}{\partial x}$, and $v\frac{\partial v}{\partial y}$, and are thus, if, as we assume, u and v are small, of the order of squares of small quantities. These momentum terms will therefore be neglected in this and the following discussions. With this stipulation the equations (13) reduce to

$$\left.\begin{aligned} \mu\frac{\partial^2 u}{\partial z^2} &= \frac{\partial p}{\partial x}, \\ \text{and} \quad \mu\frac{\partial^2 v}{\partial z^2} &= \frac{\partial p}{\partial y}. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots(14)$$

These can be directly integrated, since p is independent of z, and thus

$$\frac{\partial u}{\partial z} = \frac{1}{\mu}\frac{\partial p}{\partial x}(z + C_1),$$

$$\text{and} \quad u = \frac{1}{\mu}\frac{\partial p}{\partial x}\left(\frac{z^2}{2} + C_1 z + C_2\right),$$

$$\text{and similarly} \quad v = \frac{1}{\mu}\frac{\partial p}{\partial y}\left(\frac{z^2}{2} + D_1 z + D_2\right).$$

Now since u and v are zero on the plane $z = 0$, the integration constants C_2 and D_2 are each zero, and since u and v are also zero on the plane $z = h$:

$$\frac{h^2}{2} + C_1 h = \frac{h^2}{2} + D_1 h = 0,$$

$$\text{so that} \quad C_1 = D_1 = -\frac{h}{2}.$$

$$\text{Thus} \quad u = \frac{1}{\mu}\frac{\partial p}{\partial x}\frac{z(z-h)}{2}, \quad \ldots\ldots\ldots\ldots\ldots(15)$$

$$\text{and} \quad v = \frac{1}{\mu}\frac{\partial p}{\partial y}\frac{z(z-h)}{2}, \quad \ldots\ldots\ldots\ldots\ldots(15a)$$

and the resultant velocity of the fluid at any point is

$$(u^2 + v^2)^{\frac{1}{2}} = \frac{1}{\mu}\left\{\left(\frac{\partial p}{\partial x}\right)^2 + \left(\frac{\partial p}{\partial y}\right)^2\right\}^{\frac{1}{2}} \frac{z(z-h)}{2}, \ldots\ldots (16)$$

being in the direction of, and proportional to, the most rapid fall of pressure, and varying according to a parabolic law along each normal from one plane to the other, having its maximum value midway between them.

The total flow across a width ∂y (see fig. 9) from plane $z = 0$ to plane $z = h$ (in the direction of x increasing) is

$$\delta y \mathrm{U} = \delta y \int_0^h u dz = \frac{\delta y}{2\mu}\frac{\partial p}{\partial x}\int_0^h (z^2 - zh)dz$$

$$= \frac{\delta y}{2\mu}\frac{\partial p}{\partial x}\left[\frac{z^3}{3} - \frac{z^2 h}{2}\right]_0^h = -\delta y \frac{h^3}{12\mu}\frac{\partial p}{\partial x},$$

$$\text{or} \quad \mathrm{U} = -\frac{h^3}{12\mu}\frac{\partial p}{\partial x}. \ldots\ldots\ldots\ldots\ldots\ldots (17)$$

Similarly the flow per unit width in the y direction

$$\text{is} \quad \mathrm{V} = -\frac{h^3}{12\mu}\frac{\partial p}{\partial y}. \ldots\ldots\ldots\ldots\ldots\ldots (18)$$

Thus the total flow in any direction across a unit width perpendicular to that direction is equal to the rate of *decrease* of pressure in that direction multiplied by the constant $\frac{h^3}{12\mu}$.

The same relation evidently holds for the flow of a viscous liquid in the space between two concentric, fixed cylinders, in either the axial or the circumferential direction, provided that the radii of the cylinders are so nearly equal that their difference can be neglected compared with either of them.

In both of these cases, as well as in all other cases of flow between parallel surfaces plane or curved, it is evident, considering any small rectangular element δx, δy, which extends in the z direction from one surface to the other, that since the same amount of fluid must flow out of, as flows into, the element in unit time,

$$h\left(\frac{\partial \mathrm{U}}{\partial x} + \frac{\partial \mathrm{V}}{\partial y}\right)\delta x \delta y = 0,$$

$$\text{or from (17) and (18)} \quad \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = 0, \ldots\ldots\ldots\ldots\ldots\ldots (19)$$

it being remembered that the surfaces $z = 0$ and $z = h$, are assumed to be fixed.

Flow between Parallel Planes having Relative Motion

If the plane $z = h$ is moving parallel to the plane $z = 0$, with components of velocity u_1 and v_1 in the X and Y directions, uniform rates of shear $\frac{u_1}{h}$ and $\frac{v_1}{h}$ in these two directions will be superimposed on the fluid velocities u and v of (15) and (15*a*). The components of velocity at z will become

$$u' = \frac{1}{\mu}\frac{\partial p}{\partial x}\frac{z(z-h)}{2} + u_1\frac{z}{h},$$

$$\text{and} \qquad v' = \frac{1}{\mu}\frac{\partial p}{\partial y}\frac{z(z-h)}{2} + v_1\frac{z}{h},$$

but neither the pressures nor the relation

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = 0$$

will be affected.

If, on the other hand, the plane $z = h$ is caused to move normally away from the plane $z = 0$, with velocity $\frac{dh}{dt}$, so that the distance between the planes continually increases at this rate, it is evident that an excess of inflow over outflow must take place through the sides (at right angles to the planes) of the elementary volume $h\delta x\delta y$ to supply the additional volume which is continually being added to the element at the rate $\frac{dh}{dt}\delta x\delta y$.

Expressing this equality in symbols,

$$\delta y\frac{\partial U}{\partial x}\delta x + \delta x\frac{\partial V}{\partial y}\delta y = -\frac{dh}{dt}\delta x\delta y,$$

$$\text{or} \qquad \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = -\frac{dh}{dt},$$

and consequently from (17) and (18),

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = \frac{12\mu}{h^3}\frac{dh}{dt}. \quad \ldots\ldots\ldots\ldots\ldots (20)$$

If we take the planes as being circular, of radius a, and suppose that the fluid between them at this radius is in direct communication with a large volume of the same fluid at constant pressure Π, it is evident from symmetry that the flow will be everywhere radially inwards and that the pressure will diminish from Π at radius a to a minimum at the centre. Taking, instead of a rectangular element, a cylindrical element extending from one plane to the other and contained between radii r and $r + \delta r$ as well as between two radial planes at a small angle $\delta\alpha$ apart, its rate of increase of volume, see fig. 11, will be $\frac{dh}{dt}\delta r . r\delta\alpha$. This must be equal to the rate of increase of the inward radial flow as r increases by δr, so that from (17), p. 123,

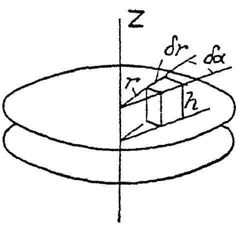


Fig. 11

$$\frac{dh}{dt}\delta r . r\delta\alpha = \frac{\partial}{\partial r}\left(\frac{h^3}{12\mu}\frac{\partial p}{\partial r} r\delta\alpha\right)\delta r,$$

$$\text{or} \quad \frac{12\mu}{h^3}\frac{dh}{dt}r = \frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right).$$

Integrating,

$$\frac{6\mu}{h^3}\frac{dh}{dt}r^2 = r\frac{\partial p}{\partial r} + \mathrm{C}.$$

But from (17), since the radial velocity is zero at the centre,

$$\frac{\partial p}{\partial r} = 0 \text{ when } r = 0,$$

$$\mathrm{C} = 0 \text{ and } \frac{\partial p}{\partial r} = \frac{6\mu}{h^3}\frac{dh}{dt}r,$$

so that integrating again

$$p = \frac{3\mu}{h^3}\frac{dh}{dt}r^2 + \mathrm{C}_1.$$

$$\text{When} \quad r = a, \ p = \Pi = \frac{3\mu}{h^3}\frac{dh}{dt}a^2 + \mathrm{C}_1,$$

$$\text{so that} \quad p = \Pi - \frac{3\mu}{h^3}\frac{dh}{dt}(a^2 - r^2). \quad \ldots\ldots(21)$$

The force, P, necessary to move the plane at $z = h$, against the

viscous resistance is equal and opposite to the difference of pressure $p - \Pi$ integrated over the whole circle, or

$$\begin{aligned} P &= \int_0^a 2\pi r \frac{3\mu}{h^3}\frac{dh}{dt}(a^2 - r^2)dr \\ &= \frac{6\pi\mu}{h^3}\frac{dh}{dt}\int_0^a (a^2 r - r^3)dr \\ &= \frac{6\pi\mu}{h^3}\frac{dh}{dt}\left(\frac{a^4}{2} - \frac{a^4}{4}\right) \\ &= \frac{3\pi\mu a^4}{2h^3}\frac{dh}{dt} \qquad \text{.....................(22)} \end{aligned}$$

Cup-and-ball Viscometer

The type of viscous motion which has just been discussed is that on which is based the action of the cup-and-ball viscometer already mentioned on p. 19, and illustrated in figs. 12 and 13. In the actual instrument, however, as illustrated in fig. 12, the two parallel surfaces which are drawn apart are not planes but segments of two spheres, one concave and the other convex. The fixed surface is the concave lower surface of a metal cup, to which is attached a hollow handle by which the instrument is suspended. In the cup fits a steel ball, but its surface is prevented from making actual contact with the spherical surface of the cup by three very small projections (J, fig. 12) from the cup's spherical surface. The two spherical surfaces are thus maintained parallel and about 0·01 mm. apart, when the ball rests on the projections. The narrow interspace is filled with the liquid to be tested, and in addition a groove G formed around the edge of the cup, and having a capacity of a few cubic millimetres, is also filled with the fluid, which is held in both the groove and the interspace by capillary tension. The groove forms the reservoir at constant pressure Π from which the interspace is fed with fluid when the two surfaces are drawn apart, as in the preceding calculations.

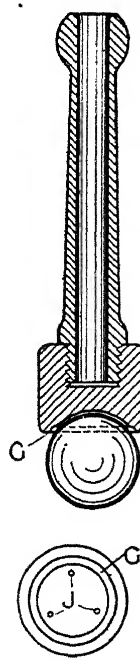


Fig. 12

The force P employed to draw the surfaces apart is the weight

of the ball, which is usually of steel and 1 inch in diameter. The method of making a test is merely, after placing sufficient liquid in

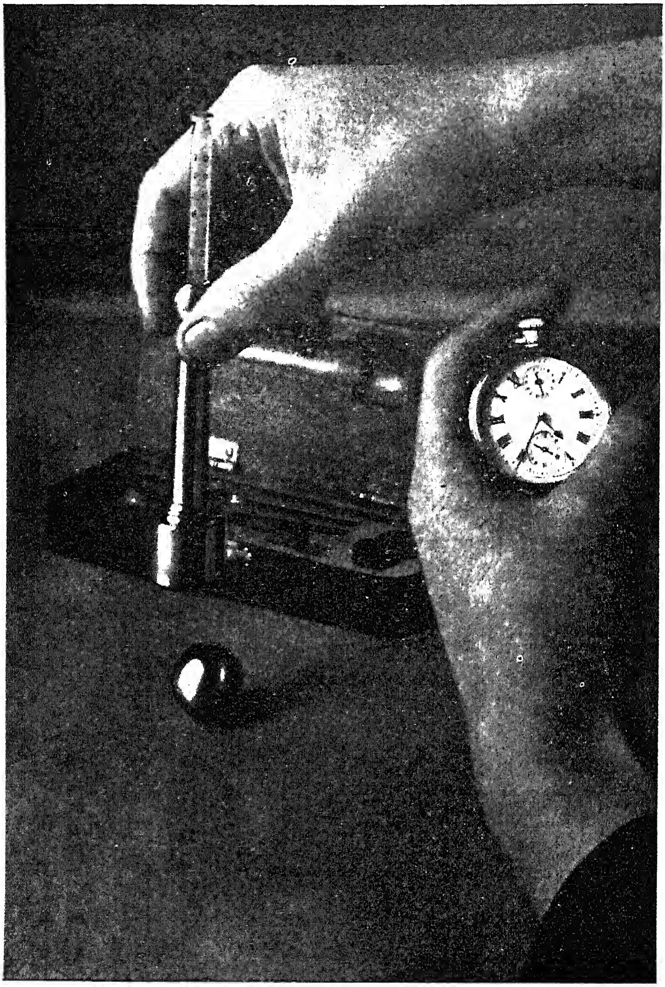


Fig. 13.—Cup-and-ball Viscometer

the cup to fill the groove and interspace, and pressing the ball home, to suspend the whole instrument and note the time by stop-watch which the ball takes to detach itself. The temperature of the instrument, which, on account of the good conductivity of the

metal and the very small mass of liquid, is also very nearly the temperature of the latter, is observed at the same time by means of a thermometer inserted in the hollow handle.

The time of fall, the dimensions of the instrument being given, can be calculated approximately from formula (22), the spherical segments concerned, which in the actual instrument are comparatively flat, being treated as circular planes of the same area.

$$\text{Since } \frac{dt}{dh} = \frac{3\pi\mu a^4}{2h^3 P},$$

$$t = \int_{h_1}^{h_2} \frac{3\pi\mu a^4}{2P}\frac{dh}{h^3} = \frac{3\pi\mu a^4}{4P}\left(\frac{1}{h_1^2} - \frac{1}{h_2^2}\right),$$

in which t is the time of fall of the ball, of weight P, from its initial distance h_1 to a final distance h_2 from the surface of the cup.

This fall is to be considered to be complete when the volume of fluid drawn into the interspace is equal to the volume initially contained in the groove, i.e.

$$\pi a^2(h_2 - h_1) = 2\pi a S$$

where S is the sectional area of the groove,

$$\text{thus } h_2 = h_1 + \frac{2S}{a},$$

and the time of the complete fall is

$$t = \frac{3\pi\mu a^4}{4P}\left\{\frac{1}{h_1^2} - \frac{1}{\left(h_1 + \frac{2S}{a}\right)^2}\right\}$$

$$= \frac{3\pi\mu a^4}{P}\,\frac{S(ah_1 + S)}{h_1^2(ah_1 + 2S)^2},$$

and if S is large compared to ah_1, as it should be,

$$t = \frac{3\pi\mu a^4}{4Ph_1^2}, \text{ and } \mu = \frac{4Ph_1^2 t}{3\pi a^4} \dots\dots\dots\dots (23)$$

It will be seen from the formula (22) that the velocity of fall $\frac{dh}{dt}$ varies as the cube of the distance fallen through. It is thus very small at first, but increases very rapidly in the later stages, and there is no difficulty in practice in deciding the moment when the fall is virtually complete.

Although the action of the cup-and-ball viscometer can be calculated with sufficient accuracy when its dimensions, including the initial thickness of the fluid film, are known, the determination of this thickness, that is to say the height of the three projections in the cup, with sufficient accuracy would be so difficult that in practice the instrument is employed as a secondary viscometer only. Each instrument requires, however, only a single calibration test, which suffices to determine a single constant for the instrument, applicable over its whole range. The corrections for the momentum of the fluid and for capillarity are negligible, the former because the velocity of the fluid is exceedingly low and the latter because the radius of curvature of the meniscus of the liquid in the groove is very large compared to the thickness of the liquid film subject to viscous traction.

B. LUBRICATION

The Connection between Lubrication and Viscosity

Although viscous liquids and plastic solids have been used from the earliest times to diminish friction between solid bodies moving in contact with one another, and although the practice of thus "lubricating" the bearings of machines has doubtless been universal since machines were first constructed, no rational explanation of the action of the lubricant was known until Osborne Reynolds (5), in 1886, gave a clear interpretation of the phenomena in terms of the theory of viscosity. Reynolds' explanation was only complete in a quantitative sense in the case of journal bearings furnished with special, and at that date unusual, means for supplying ample quantities of lubricant. He showed that in such cases the solid surfaces are completely separated from one another by fluid films of appreciable thickness, and that such films are maintained and enabled to support the pressure imposed on them quite automatically by the relative motion of the parts. The theory has since been extended to bearings of other kinds than journal bearings, and by its application new types of bearings have been devised for various purposes which have proved far more efficient than the forms which they were designed to replace.

While this "viscosity theory" of bearing lubrication is not quantitatively complete in all cases, and while there are probably other modes of lubrication in which viscosity does not play an essential part, it is at present true that all the most efficient known

types of bearings which operate with sliding, as distinguished from rolling, contact utilize the principle of lubrication which was discovered by Reynolds. The experimental and theoretical work by which the principle has been developed may be followed in the papers quoted in the bibliography attached to the end of this chapter. It is only possible in the present chapter to give an outline of the theory and a few of the leading results which have been established, with examples of the practical forms of bearings in which the theory has been utilized.

The feature common to all the bearings to which Reynolds' theory can be applied is that the surfaces of the relatively moving parts are not exactly parallel but slightly inclined to one another. For instance, in order that a journal bearing of the usual type may be lubricated according to Reynolds' principle, it is necessary that the journal shall be slightly eccentric in the bearing, so that the film of lubricant shall be of a thickness varying around the journal.

Similarly, for the proper lubrication of a slipper moving relatively to a plane surface, it is necessary that the surface of the slipper, if plane, shall be slightly inclined to the plane surface over which it moves.

Essential Condition of Viscous Lubrication

The explanation of this essential condition is readily given as an extension of the calculations contained in the first part of this chapter.

Usually in the bearings to which the theory is applicable one of the surfaces can be considered as continuous or unlimited in dimension in the direction of the relative motion (as for instance the surface of a journal, or a thrust collar, or an engine cylinder), while the surface of the other member is essentially limited or discontinuous in the same direction (as the surfaces of the corresponding bearing-brass, thrust-bearing shoe, or engine piston).

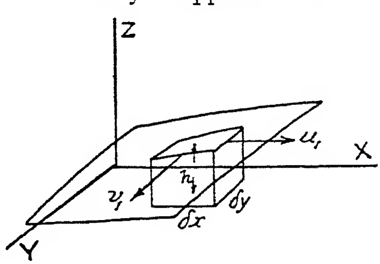


Fig. 14

In fig. 14, let XY be axes of co-ordinates (straight or curved) in directions at right angles to each other along the surface of the continuous element, and Z the

co-ordinate axis normal to this surface, i.e. in the direction of the thickness of the film, and as before let u, v, w be the components of the velocity of the fluid at any point in these three directions. The surfaces of the continuous and discontinuous elements are assumed to be nearly parallel, and the distance between them h to be small compared to their radii of curvature. The discontinuous surface is supposed to move with components of velocity u_1, v_1 in the X and Y directions, parallel to the continuous surface, at xy. The problem of finding the motions and pressures of a viscous film between the surfaces is the same as that discussed on p. 123, except that the surfaces are not now parallel. Considering, as before, the rate of change of volume and the flow of fluid into and out of an element extending from one surface to the other and standing on the base δx, δy, it is seen from equations (17), (18), that the rate of increase of volume of fluid in the element due to the rates of change of pressure and of film-thickness, in the X and Y directions is

$$\frac{\partial}{\partial x}\left(\frac{h^3}{12\mu}\frac{\partial p}{\partial x}\right)\delta x\delta y + \frac{\partial}{\partial y}\left(\frac{h^3}{12\mu}\frac{\partial p}{\partial y}\right)\delta y\delta x, \quad \ldots\ldots\ldots(24)$$

while the rate at which fluid passes out of the element in consequence of the shearing deformation due to the movement of the upper surface over the lower is

$$\frac{u_1}{2}\frac{\partial h}{\partial x}\delta x\delta y + \frac{v_1}{2}\frac{\partial h}{\partial y}\delta y\delta x. \quad \ldots\ldots\ldots\ldots(25)$$

The volume of the element is however diminishing, in consequence of the movement of the upper plane, at the rate

$$u_1\frac{\partial h}{\partial x}\delta x\delta y + v_1\frac{\partial h}{\partial y}\delta y\delta x,$$

consequently

$$\frac{\partial}{\partial x}\left(\frac{h^3}{12\mu}\frac{\partial p}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{h^3}{12\mu}\frac{\partial p}{\partial y}\right) - \left(\frac{u_1}{2}\frac{\partial h}{\partial x} + \frac{v_1}{2}\frac{\partial h}{\partial y}\right) = -\left(u_1\frac{\partial h}{\partial x} + v_1\frac{\partial h}{\partial y}\right),$$

or $$\frac{\partial}{\partial x}\left(h^3\frac{\partial p}{\partial x}\right) + \frac{\partial}{\partial y}\left(h^3\frac{\partial p}{\partial y}\right) + 6\mu\left(u_1\frac{\partial h}{\partial x} + v_1\frac{\partial h}{\partial y}\right) = 0 \ldots\ldots\ldots(26)$$

This is the general differential equation determining the value of p at every point, being solved by integration for each particular case when h is given as a function of x and y (thus defining the forms of the surfaces), and when the velocities u_1, v_1 are assigned. The complete solution is often not practicable, but exact or approximate

solutions can be obtained in a number of the simpler cases which can be regarded as sufficiently close approximations to the actual conditions of various types of bearings.

Inclined Planes Unlimited in one Direction

Take the case of two plane surfaces, the lower, $z = 0$, being unlimited in the directions of X and Y, while the upper, also unlimited in the direction of Y,* extends only from $x = a_1$, to $x = a_2$, and intersects the plane $z = 0$ on the line $x = 0$. Thus the distance between the planes, everywhere small, is proportional to x, so that $h = cx$, where c is the tangent of the small angle between the planes (see fig. 15). Let us assume that the upper plane moves over the lower with velocity u_1, in the direction of X, v being zero, and that the whole is immersed in fluid, so that the pressure both in front of and behind the moving plane is Π and is constant. Obviously none of the conditions vary in the direction of Y, so that $\frac{\partial h}{\partial y}$ and $\frac{\partial p}{\partial y}$ are both zero. Thus equation (26) becomes

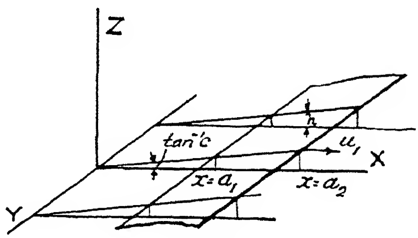


Fig. 15

$$\frac{\partial}{\partial x}\left(h^3\frac{\partial p}{\partial x}\right) + 6\mu u_1\frac{\partial h}{\partial x} = 0,$$

and therefore $$h^3\frac{\partial p}{\partial x} + 6\mu u_1(h - h_1) = 0, \quad \text{.............(27)}$$

h_1 being the value of h where $\frac{\partial p}{\partial x} = 0$, that is to say at a point, say x_1, where p has a maximum or minimum value

Thus $$\frac{\partial p}{\partial x} = -6\mu u_1\left(\frac{1}{h^2} - \frac{h_1}{h^3}\right) = -\frac{6\mu u_1}{c^2}\left(\frac{1}{x^2} - \frac{x_1}{x^3}\right).$$

Since $\frac{\partial p}{\partial x}$ is positive when $h < h_1$, and negative when $h > h_1$, it is seen

* The dimension of a bearing in the direction of the motion will in all cases be referred to as its length, and the transverse dimension as its width, regardless of which of these is the greater.

that p has a maximum value (at $x = x_1$), between $x = a_1$, and $x = a_2$.

Integrating, $$p = \frac{6\mu u_1}{c^2}\left(\frac{1}{x} - \frac{x_1}{2x^2} - C\right), \qquad \text{(28)}$$

but since $p = \Pi$, both when $x = a_1$, and when $x = a_2$,

$$\Pi = \frac{6\mu u_1}{c^2}\left(\frac{1}{a_1} - \frac{x_1}{2a_1^2} - C\right) = \frac{6\mu u_1}{c^2}\left(\frac{1}{a_2} - \frac{x_1}{2a_2^2} - C\right),$$

from which

$$\frac{x_1}{2}\left(\frac{1}{a_1^2} - \frac{1}{a_2^2}\right) = \left(\frac{1}{a_1} - \frac{1}{a_2}\right),$$

$$\text{or} \qquad x_1 = \frac{2a_1a_2}{a_1 + a_2} \qquad \text{(29)}$$

(the point of maximum pressure thus being nearer to a_1 than to a_2),

$$\text{and} \quad -\frac{6\mu u_1 C}{c^2} = \Pi - \frac{3\mu u_1}{c^2}\left\{\frac{1}{a_1} + \frac{1}{a_2} - \frac{x_1}{2}\left(\frac{1}{a_1^2} + \frac{1}{a_2^2}\right)\right\}$$

$$= \Pi - \frac{6\mu u_1}{c^2(a_1 + a_2)}.$$

Thus by substitution for x_1, and C in (28),

$$p = \Pi + \frac{6\mu u_1}{c^2}\left(\frac{1}{x} - \frac{a_1a_2}{(a_1 + a_2)x^2} - 1\right)$$

$$= \Pi + \frac{6\mu u_1}{c^2(a_1 + a_2)}\left(\frac{a_1 + a_2}{x} - \frac{a_1a_2}{x^2} - 1\right), \qquad \text{(30)}$$

this equation determining the pressures at all points between the two planes.

The total upward pressure on the upper plane per unit width in the direction Y is

$$P = \int_{a_1}^{a_2}(p - \Pi)dx = \frac{6\mu u_1}{c^2(a_1 + a_2)}\int_{a_1}^{a_2}\left(\frac{a_1 + a_2}{x} - \frac{a_1a_2}{x^2} - 1\right)dx$$

$$= \frac{6\mu u_1}{c^2(a_1 + a_2)}\left[(a_1 + a_2)\log_e x + \frac{a_1a_2}{x} - x\right]_{a_1}^{a_2}$$

$$= \frac{6\mu u_1}{c^2(a_1 + a_2)}\left\{(a_1 + a_2)\log_e\frac{a_2}{a_1} + 2(a_1 - a_2)\right\}$$

$$= \frac{6\mu u_1}{c^2}\left\{\log_e\frac{a_2}{a_1} + 2\frac{a_1 - a_2}{a_1 + a_2}\right\}, \qquad \text{(31)}$$

being dependent only on the ratio of a_2 to a_1, for a given value of c, and the mean pressure is

$$\frac{P}{a_2 - a_1} = \frac{6\mu u_1}{c^2}\left\{\frac{1}{a_2 - a_1}\log_e\frac{a_2}{a_1} - \frac{2}{a_2 + a_1}\right\} \quad \ldots\ldots(32)$$

Also the total frictional resistance to the motion of the upper plane per unit width is

$$F = \int_{a_1}^{a_2}\mu\frac{u_1}{h}dx = \int_{a_1}^{a_2}\frac{\mu u_1}{cx}dx \quad \ldots\ldots\ldots\ldots\ldots(33)$$

$$= \frac{\mu u_1}{c}\log_e\frac{a_2}{a_1}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(33a)$$

dependent, like P, only on the ratio $a_2 : a_1$ and c; and the ratio of traction to load, or "coefficient of friction" is

$$f = \frac{F}{P} = \frac{c}{6}\frac{\log_e\dfrac{a_2}{a_1}}{\log_e\dfrac{a_2}{a_1} - 2\dfrac{a_1 - a_2}{a_1 + a_2}}. \quad \ldots\ldots\ldots\ldots(34)$$

Also the position of the centre of the upward pressure on the upper plane is given by

$$\bar{a} = \frac{1}{P}\int_{a_1}^{a_2}(p - \Pi)x\,dx = \frac{6\mu u_1}{Pc^2(a_1 + a_2)}\int_{a_1}^{a_2}\left\{(a_1 + a_2) - \frac{a_1 a_2}{x} - x\right\}dx$$

$$= \frac{6\mu u_1}{Pc^2(a_1 + a_2)}\left\{\frac{a_1^2 - a_2^2}{2} - a_1 a_2\log_e\frac{a_2}{a_1}\right\}$$

$$= \frac{1}{2}\,\frac{a_2^2 - a_1^2 - 2a_1 a_2\log_e\dfrac{a_2}{a_1}}{2(a_1 - a_2) + (a_1 + a_2)\log_e\dfrac{a_2}{a_1}}, \quad \ldots\ldots\ldots\ldots\ldots\ldots(35)$$

being independent of c.

Applications to Actual Bearings

The solution of the problem in viscous motion illustrated in fig. 15 has been worked out in some detail because it affords in a single case a general view of the nature of Reynolds' theory of lubrication.

If we imagine the lower plane $z = 0$ replaced by the surface of a cylinder whose axis is parallel to the Y axis of co-ordinates, and

the upper plane, extending from $x = a_1$, to $x = a_2$, replaced by a curved surface which, at every point of co-ordinates x, y, measured respectively circumferentially from a generating line of the cylinder corresponding to $x = 0$, and axially from a circumferential circle of the cylinder corresponding to $y = 0$, is at the same normal distance h from the cylinder as are the two planes from one another, the results which have been obtained will still apply. This ideal form of a cylindrical journal bearing is illustrated in fig. 16. The cylinder can be regarded as the journal of an axle, and the upper surface as the bearing surface of the bearing-brass of the axle.

The results as to the fluid pressure which have been calculated above evidently remain true if, instead of the bearing-brass moving in the direction x with linear velocity u_1, the journal revolves in the opposite direction with the same surface velocity.

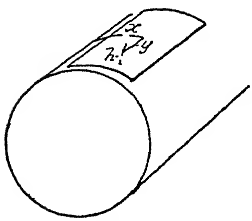


Fig. 16

Actual bearings are, of course, not of unlimited width, but for the middle portions of a bearing whose dimension in the direction transverse to the relative motion is not less than two or three times that in the direction of motion, the calculated results apply with fair accuracy. In such middle portions of the bearing the oil will flow in lines approximately at right angles to the generating lines of the cylinder. In the lateral portions of the bearing, on the other hand, the oil being under pressure will tend to flow towards the nearest side, and the theoretical conditions will on this account be departed from. If, however, the sides of the bearing be closed by some arrangement, such as a stuffing-box, preventing the escape of oil, the flow of oil will be everywhere, except within distances from the closed sides comparable with h_1, circumferential, and the conditions assumed for unlimited surfaces will be precisely realized, provided always, of course, that the bearing-brass is of such a form that $h = cx$, which is true only to a first approximation for the form which is usually given to such brasses.

The calculations apply more accurately to the case of a conical sleeve moving longitudinally on a cylindrical rod as illustrated in fig. 17. In this figure the axis of X is a generating line of the cylindrical surface of the rod, the axis of Y is a circumferential circle, and that of Z as before is normal to the surface. As before, we assume that the normal distance between the surfaces is given by $h = cx$,

so that the conical and cylindrical surfaces, which are coaxial, intersect at $x = 0$. The sleeve extends from $x = a_1$ to $x = a_2$, and is supposed to move parallel to the axis of X with velocity u_1.

From symmetry the motion of the fluid must be everywhere parallel to the axis of X, and as the cone and film of fluid have no boundaries in the direction of Y, the solution given above will hold accurately provided that the thickness of the film is very small compared to the radius r_1, and length, $a_2 - a_1$, of the cone. Thus, for example, the resistance to the motion of the cone, from (33*a*), p. 133, is

$$2\pi r_1 F = \frac{2\pi r_1 \mu u_1}{c} \log_e \frac{a_2}{a_1}.$$

The curve p_1 in fig. 18 shows the mode in which the fluid pressure

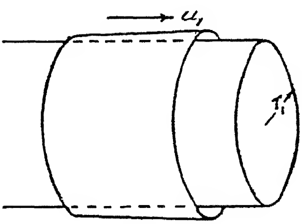


Fig. 17

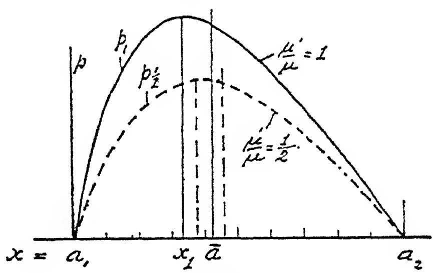


Fig. 18

between the surfaces of figs. 15, 16, and 17 varies in the direction of x for the particular case in which $a_2 = 2a_1$. It will be seen that the maximum pressure occurs at $x_1 = \frac{4}{3}a_1$, or at one-third of the length of the sleeve or bearing-brass from its rear end, and, as may be seen by writing $2a_1$ for a_2 in (35), the resultant pressure occurs at $\bar{a} = 1{\cdot}431 a_1$, or $0{\cdot}431$ of the length from the same end.

Table III, p. 130, shows the actual numerical results in C.G.S. units for a moving surface carrying a resultant pressure of 1 Kgm. with a lubricating fluid of viscosity 1 C.G.S. The surface is assumed to be 1 cm. long in the direction of motion (i.e. $a_2 - a_1 = 1$ cm.), and the results are expressed for 1 cm. of width in the transverse direction. The quantities tabulated are:

h_1, the thickness of film at $x = a_1$, unit 10^{-3} cm.;
h_2, the thickness of film at $x = a_2$, unit 10^{-3} cm.;

$\bar{a} - a_1$, the distance of the centre of pressure from the trailing end.

Unit, 1 cm.; f, the effective coefficient of friction, $= F_1$ the tractive force in kilograms.

The independent variable in the first column of the table is the ratio $\frac{a_2}{a_1} = \frac{h_2}{h_1}$.

TABLE III

$\frac{a_2}{a_1}$.	h_1.	h_2.	$\bar{a} - a_1$.	f.
1·0	0	0	0·5000	∞
1·2	0·2775	0·3330	0·4818	$3·285 \times 10^{-3}$
1·4	0·3465	0·4851	0·4664	2·428
1·6	0·3793	0·6079	0·4532	2·065
1·8	0·3955	0·7119	0·4416	1·858
2·0	0·4026	0·8051	0·4313	1·722
2·2	0·4043	0·8895	0·4221	1·625
2·4	0·4027	0·9665	0·4137	1·553
2·6	0·3991	1·0375	0·4061	1·497
2·8	0·3942	1·1037	0·3991	1·451
3·0	0·3884	1·1652	0·3926	1·414
4·0	0·3559	1·4237	0·3662	1·298
5·0	0·3247	1·6237	0·3465	1·239
6·0	0·2982	1·7892	0·3310	1·202
11·0	0·2115	2·3269	0·2832	1·134

The corresponding results for any other dimensions and conditions of loading may be derived from the following dimensional formulæ, viz.:

If the length of the surface, velocity, resultant load, and viscosity, instead of being each unity in the units employed, are respectively

Length, L centimetres,
Velocity, V centimetres per second,
Load, P kilograms per unit width,
Viscosity, M C.G.S. units,

then, for any given value of $\frac{a_2}{a_1}$, h_1 and h_2 are to be multiplied by $\frac{LV^{\frac{1}{2}}M^{\frac{1}{2}}}{P^{\frac{1}{2}}}$, $\frac{\bar{a} - a_1}{a_2 - a_1}$ is unchanged, and F is to be multiplied by $V^{\frac{1}{2}}P^{\frac{1}{2}}M^{\frac{1}{2}}$, while c and f are to be multiplied by $\frac{M^{\frac{1}{2}}V^{\frac{1}{2}}}{P^{\frac{1}{2}}}$.

It will be seen from Table III, combined with these dimensional formulæ, that the thicknesses of the films of viscous fluid concerned

in lubrication are small, and comparable to the smallest linear measurements which the mechanical engineer is accustomed to make. It is therefore necessary in order to effect lubrication in the manner intended, and to secure the low frictional resistances which the theory indicates as attainable, that the workmanship of the bearings shall be of a relatively high order of accuracy.

The fact, otherwise inexplicable, that the conditions and laws of viscous lubrication were not discovered until the end of the nineteenth century, is doubtless due to the circumstance that it was only at about that epoch that mechanical workmanship became generally of such a quality that the necessary conditions were often complied with. With rougher workmanship the necessary continuous films cannot be formed, but the two members of the bearing come into actual or virtual contact, at least at some points, and thus bring about mixed conditions of solid and viscous friction incapable of being referred to any simple or consistent laws.

Even with workmanship which may be regarded as perfect the ultimate stage of failure initiated by any cause is contact of the solid surfaces, either directly or through the small particles of solid impurities which are always to some extent present in the lubricant. There is thus suggested as a criterion of the safety of any bearing from such failure, the thickness of the lubricating film at its thinnest part under the working conditions which reduce this thickness to a minimum.

It will be seen from Table III, p. 136, that for a bearing surface of given length, with given velocity, load, and lubricant, the thickness of the lubricant at the point of closest approach to the other surface, is a maximum when $\frac{a_2}{a_1} = 2{\cdot}2\ldots$ It is usual to adopt this ratio as that to be preferred in designing bearings. The table shows that with this ratio the coefficient of friction, though higher than is attainable with greater values of the ratio $\frac{a_2}{a_1}$, is nevertheless already so small that its further reduction may usually be considered of little moment. It must, however, be remembered that the optimum ratio, $\frac{a_2}{a_1} = 2{\cdot}2$ (often taken as $2{\cdot}0$ as a sufficiently close approximation) has, strictly speaking, been derived only from the special case of a bearing surface of infinite length and for the condition $h = cx$.

It is hardly necessary to remark that if the velocity u_1 in the above calculations be reversed, the equations for the pressures will be still

valid, with merely a change of sign for both u and p. It must, however, be remembered that whereas in the case of positive values of p the intensity of pressure has no necessary limit, negative values of p, that is to say tensions, are not in general sustainable in fluids such as ordinary oils, and indeed in most forms of bearings positive values of p less than Π, the atmospheric pressure, are usually inconsistent with the assumptions made in the calculations, since, under those conditions, air will be drawn into the spaces assumed to be occupied by oil.

The volume of fluid flowing between the surfaces per unit time may be calculated as follows:

From (30), p. 132, the rate of change of the pressure with x at the rear end of the bearing, i.e. at $x = a_1$, is

$$\frac{\partial p}{\partial x} = -\frac{6\mu u_1}{c^2}\left[\frac{1}{a_1^2} - \frac{2a_2}{a_1^2(a_1+a_2)}\right]$$

$$= -\frac{6\mu u_1}{c^2 a_1^2}\left[\frac{a_1 - a_2}{a_1 + a_2}\right].$$

Therefore from (17), p. 122, and (25), p. 130, the volume rate at which the fluid passes through unit width of the normal plane at the rear end of the moving surface is

$$Q = -\frac{(ca_1)^3}{12\mu}\frac{6\mu u_1(a_1 - a_2)}{c^2(a_1+a_2)a_1^2} + \frac{ca_1}{2}u_1$$

$$= \frac{ca_1u_1}{2}\left(1 - \frac{a_1 - a_2}{a_1 + a_2}\right) = \frac{cu_1a_1a_2}{a_1 + a_2}. \quad \ldots\ldots(36)$$

The same result would be obtained by calculating the inflow at the front edge, and it may also be seen at once from the consideration that at the point of maximum pressure $x = x_1$, there being no flow due to rate of change of pressure, the volume rate at which fluid passes the normal plane is entirely due to the mean velocity $\frac{u_1}{2}$, acting over the film thickness, which is

$$cx_1 = \frac{2ca_1a_2}{a_1 + a_2}.$$

Under the same assumptions as in Table III, p. 136, the value of Q for the condition $\frac{a_2}{a_1} = 2{\cdot}2$ is $2{\cdot}78 \times 10^{-4}$ c. c. per second per centimetre of the transverse dimension of the bearing.

Self-adjustment of the Positions of Bearing Surfaces

The question naturally arises how it is possible to secure in actual bearings the exact locations of the bearing parts shown to be necessary by the preceding calculations, and as illustrated in figs. 15–17, and how it is that so delicate an adjustment is not liable to be destroyed by inevitable wearing of the parts. The explanation is that in successful types of bearings the parts are self-adjusting, their correct mutual location being automatically brought about by their relative motion and continually corrected for any slight wear which may take place.

Take for instance the case of the infinite plane slipper illustrated in fig. 15, of which fig. 19 is a section on any plane parallel to XZ.

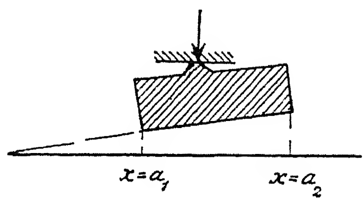


Fig. 19

It has been seen from Table III that if the ratio of a_2 to a_1 is 2·2 the resultant pressure of the fluid acts at the point $x = \bar{a}$, where $\bar{a} - a_1 = 0{\cdot}4221 \times (a_2 - a_1)$, and that with the value of h_1 given in the table and unit values of μ and u_1 the total resultant pressure is 1 Kgm. per unit transverse width. Conversely, if a load of 1 Kgm. per unit width be applied to the slipper at the point $x = \bar{a}$ as indicated by the arrow in fig. 19, and the slipper be moved with unit velocity and supplied with fluid of viscosity 1 C.G.S., it will take up the same position. Experience, moreover, shows that such equilibrium is stable for the displacements which are liable to occur in the operation of the bearings. In the case of plane slippers the load must in practice be applied as shown in fig. 19, that is to say, through an actual or virtual pivot of some kind with which the slipper is provided at the correct point. Actual examples will be illustrated in the descriptions of thrust bearings given in the later parts of this chapter.

In the case of cylindrical journal bearings, however, there is another mode of self-adjustment possible, which, though not so efficient as the pivot method, is even simpler, and which undesignedly took place in bearings of this class long before Reynolds' principle was discovered, and rendered them superior in efficiency to all other classes of bearings known at that time.

Self-adjustment in Journal Bearings

This action is illustrated for the ordinary form of fixed journal bearings in figs. 20*a*, *b*, *c*. We will assume that the bearing is one of a pair of journal bearings, as for the shaft of an electric motor, consisting of a cylindrical brass, or pair of semi-cylindrical half-brasses, of which only the lower half cylinder is normally effective. The radius of the bearing is, necessarily, greater than that of the journal. The load W is assumed to be the weight of the shaft and parts attached to it, acting vertically downwards.

When the journal is at rest its position in the bearing is that

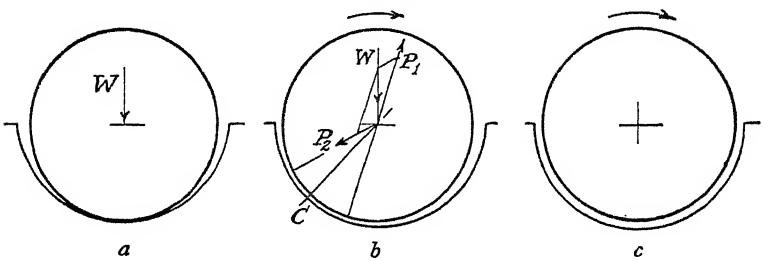


Fig. 20

shown in fig. 20*a*. The journal and bearing are then in contact along the lowest generating lines of their cylindrical surfaces. When, however, the shaft begins to rotate, for example, in the clockwise direction as indicated in the figures, the oil at the right-hand side of the journal is subjected to a traction directed from the wider to the narrower part of the interspace between the journal and the bearing. On the principles which have been explained, the oil in this space will consequently exert a fluid pressure. On the opposite, or left-hand side of the journal, on the contrary, the interspace increases in thickness in the direction of motion, and consequently, as explained on p. 137, the pressure in the oil film will fall, becoming negative unless, as is sometimes the case, air is free to enter, when atmospheric pressure will tend to be established. The journal will consequently tend to move towards this left-hand side, the point of contact between journal and bearing shifting from the lowest generating lines to some higher line towards the left hand. Oil under pressure will thus be admitted between the parts of journal and bearing, and this action will be progressive until the resultant upward pressure becomes equal to the load W on the bearing. At

constant speed a stable condition will be reached as shown in fig. 20*b*. The point of closest approach, C, will be somewhere on the left-hand side of the vertical, with a portion of the interspace above and to the left of C still diverging in the direction of motion. The oil in this latter space will in consequence exert a negative pressure on the journal, as indicated by the arrow P_2. The resultant of this force and the positive resultant pressure P_1, exerted by the oil in the right-hand converging portion of the interspace, will be equal and opposite to W, the load on the journal.

If the speed of the journal is increased, the amount of convergence and divergence of the respective parts of the journal, for a given load W, will automatically diminish, the limiting condition with infinite speed (or zero load) being that illustrated in fig. 20*c*, the journal becoming then concentric with the bearing.

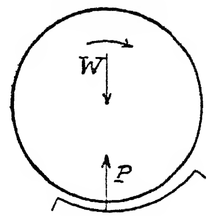


Fig. 21

It may be noticed that in all cases the diverging portion of the film, and the nearly parallel portions in the immediate neighbourhood of C, though of respectively negative and zero value for the support of load, are subject to shear of equal or greater intensity than the effective pressure-producing film on the right hand and lower surfaces of the journal. For this reason such journal bearings, with the brasses embracing a semicircle or other relatively large arc, are decidedly inefficient compared to a pivoted bearing of small arc such as that illustrated in fig. 16, in which self-adjustment takes place in the same mode as that described in connection with fig. 19.

It is also readily seen that, in all cases, the interspace between the journal and a segmental cylindrical bearing surface can only be convergent throughout its length if the arc of the bearing surface is less than 90°. It is indeed desirable, in order to secure a fairly rapid rate of convergence throughout, that the arc should be limited to 45° at most.

It will be seen that in such a case as that illustrated in fig. 21, it is possible, without pivoting the brass, for the resultant fluid pressure to be vertical and thus in equilibrium with the load W, without the formation of any diverging interspace, and this even when the radius of curvature of the brass is the same as that of the journal. The latter is a convenient condition, as it admits of the simplest and most accurate method of accurately forming the bearing surface, namely by scraping or lapping it.

It is to be remarked, however, that automatic self-adjustment in journal bearings with rigid (i.e. non-pivoted) brasses, as in fig. 20*b* or 21, is only possible when there are not more than two bearings on a shaft, or if the shaft is flexible, as otherwise, since a third bearing will be invariably out of alignment to an extent comparable with or greater than the thickness of effective oil-films, it is not possible for each of the journals to adjust itself to its correct position. With any number of pivoted bearings, however, if the pivots are approximately in the vertical plane through the axis of the shaft, each bearing will exert a vertical resultant pressure, and by providing adjustments for the pivots in the vertical direction only it is possible to divide the total load carried by the shaft equally between the bearings.

Exact Calculation of Cylindrical Journal and Bearing

The mathematical solution of the viscous motion for the case illustrated in figs. 20*a*, 21 was given by Reynolds (5, p. 158). The solution was simplified by Sommerfeld (8, p. 158), of whose process a brief résumé will now be given. In fig. 22, O and O′ are the centres and r and $r + \delta$ the radii of the cylindrical journal and semi-cylindrical bearing, both of infinite extension in the direction of their axes.

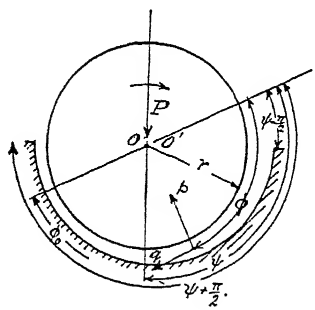


Fig. 22

Let $OO' = \epsilon$, and $\frac{\delta}{\epsilon} = \alpha$, α having therefore different values in different cases, varying from 1, when the journal and bearing are in contact, to ∞ when they are concentric.

Let ψ be the angle between OO′ and the vertical and ϕ an angular co-ordinate measured from the direction OO′, the co-ordinates for the ends of the bearing-brass being $\psi - \frac{\pi}{2}$, and $\psi + \frac{\pi}{2}$ as indicated in the figure. Thus the linear co-ordinate, x, in the direction of motion of the brass relatively to the journal is now *constant* $- r\phi$. Then, from (27), p. 131,

$$\frac{dp}{rd\phi} = 6\mu u_1 \frac{h - h_1}{h^3}, \ldots\ldots\ldots\ldots\ldots (37)$$

and since $p = \Pi$ when $\phi = \psi - \frac{\pi}{2}$, and when $\phi = \psi + \frac{\pi}{2}$,

$$\int_{\psi-\frac{\pi}{2}}^{\psi+\frac{\pi}{2}} \frac{dp}{d\phi} d\phi = 6\mu u_1 r \int_{\psi-\frac{\pi}{2}}^{\psi+\frac{\pi}{2}} \frac{h-h_1}{h^3} d\phi = 0.$$

Also, if p is the fluid pressure and q the circumferential traction per unit width at ϕ, and P the total load on the bearing per unit width,

$$\mathrm{P}\cos\psi - \int_{\psi-\frac{\pi}{2}}^{\psi+\frac{\pi}{2}} (p-\Pi)\cos\phi r d\phi - \int_{\psi-\frac{\pi}{2}}^{\psi+\frac{\pi}{2}} q \sin\phi r d\phi = 0,$$

and

$$\mathrm{P}\sin\psi - \int_{\psi-\frac{\pi}{2}}^{\psi+\frac{\pi}{2}} (p-\Pi)\sin\phi r d\phi + \int_{\psi-\frac{\pi}{2}}^{\psi+\frac{\pi}{2}} q \cos\phi r d\phi = 0.$$

But since

$$\int_{\psi-\frac{\pi}{2}}^{\psi+\frac{\pi}{2}} (p-\Pi)\cos\phi d\phi = \Big[(p-\Pi)\sin\phi\Big]_{\psi-\frac{\pi}{2}}^{\psi+\frac{\pi}{2}} - \int_{\psi-\frac{\pi}{2}}^{\psi+\frac{\pi}{2}} \frac{dp}{d\phi}\sin\phi d\phi,$$

and

$$\int_{\psi-\frac{\pi}{2}}^{\psi+\frac{\pi}{2}} (p-\Pi)\sin\phi d\phi = -\Big[(p-\Pi)\cos\phi\Big]_{\psi-\frac{\pi}{2}}^{\psi+\frac{\pi}{2}} + \int_{\psi-\frac{\pi}{2}}^{\psi+\frac{\pi}{2}} \frac{dp}{d\phi}\cos\phi d\phi,$$

and since $p = \Pi$, both when $\phi = \psi + \frac{\pi}{2}$ and when $\phi = \psi - \frac{\pi}{2}$, so that the terms not under integral signs vanish,

$$\left.\begin{aligned} &\int_{\psi-\frac{\pi}{2}}^{\psi+\frac{\pi}{2}} \left(\frac{dp}{d\phi} - q\right)\sin\phi d\phi = -\frac{\mathrm{P}}{r}\cos\psi, \\ \text{and} \quad &\int_{\psi-\frac{\pi}{2}}^{\psi+\frac{\pi}{2}} \left(\frac{dp}{d\phi} - q\right)\cos\phi d\phi = +\frac{\mathrm{P}}{r}\sin\psi. \end{aligned}\right\} \quad \ldots\ldots\ldots(38)$$

Now from (33), p. 133, and (37), p. 142,

$$\frac{1}{r}\left(\frac{dp}{d\phi} - q\right) = 6\mu u_1 \frac{h-h_1}{h^3} + \frac{\mu u_1}{rh},$$

in which the second term on the right can be neglected on account of the smallness of h compared to r.

Thus the equations can be written

$$\left.\begin{aligned} 6\mu u_1 r\int_{\psi-\pi/2}^{\psi+\pi/2}\{(h-h_1)/h^3\}\sin\phi d\phi &= -(\mathrm{P}/r)\cos\psi, \\ 6\mu u_1 r\int_{\psi-\pi/2}^{\psi+\pi/2}\{(h-h_1)/h^3\}\cos\phi d\phi &= (\mathrm{P}/r)\sin\psi, \end{aligned}\right\} \quad . . (39)$$

or, since $h = \epsilon(a + \cos\phi)$, $h_1 = \epsilon(a + \cos\phi_1)$,

these equations become

$$\left.\begin{aligned} \int\frac{\sin\phi}{(a+\cos\phi)^2}d\phi - \frac{h_1}{\epsilon}\int\frac{\sin\phi}{(a+\cos\phi)^3}d\phi &= -\frac{\delta^2\mathrm{P}\cos\psi}{6\mu a^2 r^2 u_1}, \\ \int\frac{\cos\phi}{(a+\cos\phi)^2}d\phi + \frac{h_1}{\epsilon}\int\frac{\cos\phi}{(a+\cos\phi)^3}d\phi &= \frac{\delta^2\mathrm{P}\sin\psi}{6\mu a^2 r^2 u_1}, \end{aligned}\right\} \quad . . (40)$$

the integrals as before being from $\psi - \frac{\pi}{2}$ to $\psi + \frac{\pi}{2}$.

These integrations can be effected by usual methods,* and from the results Sommerfeld calculated the following numerical table, Table IV, in which $\eta_0 = \frac{\delta}{r}$, and the " coefficient of friction ", $f = \frac{\mathrm{M}}{\mathrm{P}r}$, where M is the moment of the frictional tractions about O.

TABLE IV

a^2.	ψ.	$\cos\phi_0$.	u_1.	f.
1	90°	− 1·0	0	$\eta_0 \times$ 1·00
1·02	120°	− 0·998	$\frac{\mathrm{P}\eta_0^2}{\mu} \times$ 0·012	× 0·94
1·13	129°	− 0·98	× 0·04	× 0·93
1·5	135°	− 0·93	× 0·08	× 0·92
2·4	133°	− 0·88	× 0·14	× 1·00
6·3	128°	− 0·72	× 0·29	× 1·34
33·9	120°	− 0·50	× 0·62	× 2·17
∞	90°	0	× ∞	× ∞

If the coefficient of friction f be plotted with u_1 as the variable,

$$^*\int\frac{\cos\phi}{a+\cos\phi}d\phi = \int\frac{(a+\cos\phi)-a}{a+\cos\phi}d\phi = \phi - a\int\frac{d\phi}{a+\cos\phi}$$
$$= \phi - a\frac{2}{\sqrt{a^2-1}}\tan^{-1}\left(\tan\frac{\phi}{2}\sqrt{\frac{a-1}{a+1}}\right).$$

Differentiate this with respect to a to get integrals in second line of (40). Integrals in first line come at once, since $d(a + \cos\phi) = -\sin\phi d\phi$.

P being constant, then for various values of $\eta_0 = \frac{\delta}{r}$ we have a series of curves, in which as u_1 increases from zero, the coefficient of friction falls at first to a minimum value about 8 per cent lower than its initial value, then with further increase of u_1 the coefficient gradually rises and finally increases to an asymptotic approximation to the straight line $f = \frac{\mu\pi r u_1}{P\delta}$.

The value of u_1 for which the coefficient of friction is a minimum is approximately

$$u_0 = \frac{P\delta^2}{12{\cdot}5 \times \mu r^2}.$$

In actual bearings the initial value of the coefficient of friction will be much higher than that calculated, since with very low velocities, and values of α only slightly greater than 1, the journal and bearing will be, owing to minute roughnesses of their surfaces, in metallic contact instead of being separated by a very thin continuous film, as assumed in the theory.

It is to be observed that in these calculations of Sommerfeld's the portion of the film between the point of closest approach, $\phi = \phi_0$, and $\phi = \psi + \frac{\pi}{2}$ is subject to a negative pressure. The possibility of such a condition may reasonably be postulated in very wide bearings, but can hardly be assumed in bearings of usual proportions unless special means are employed for preventing the entry of air at the sides.

Approximate Calculation of Cylindrical Bearings

The method and results of Sommerfeld's investigation given above apply to the case of a cylindrical bearing whose angular length in the circumferential direction is 180°. A similar process may be applied to bearings of smaller angular length, as in fig. 21, such bearings, as explained on p. 141, being preferable in practice. In these cases, however, there is little value in the assumption that the surface of the brass is a circular cylinder, and, especially in the pivoted type, it is usually sufficient to assume that the thickness of the interspace is a linear function of ϕ, that is to say to apply in the case of a very wide bearing the method and results of pp. 131-133.

If a closer approximation is desired, the form of the bearing may

be approximately represented by the equation $h = cx^m = C(r\phi)^m$ with an appropriate value of m differing from unity. The solution of this case has been given by Rayleigh (20, p. 159), who, however, found that in the numerical applications which he made of it, the results did not differ very materially from those derived from the simpler formula, $h = cx$.

Plane Bearings of Finite Width

A more important modification of the Reynolds' theory of bearings with uniformly varying interspaces, is that which it requires for its application to bearings which are of limited width, and in which, consequently, there is a transverse flow of the fluid under pressure to the sides, i.e. in the direction of Y, as well as flow in the direction of the relative motion X.

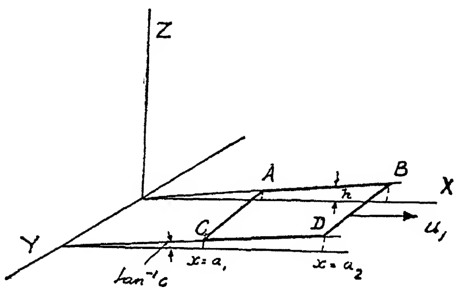


Fig. 23

The solution, as given by Michell (9, p. 158), involves rather lengthy calculations, and we can give only an indication of the method and a few working formulæ and constants.

In fig. 23 (which corresponds to fig. 15 for the case of infinite width), ABDC is a rectangular plate in the plane $z = cx$ (the length of the plate being $a_2 - a_1$, and its width b) sliding in the direction of X with velocity u_1.

The pressure is assumed to be uniform everywhere except in the interspace between the plate ABDC and the infinite fixed plate in the plane $z = 0$, i.e. the boundary conditions of the plate are $p = \Pi$, when $x = a_1$, or $x = a_2$, for all values of y, and also when $y = 0$, or $y = b$, for all values of x. Between the two plates p must satisfy the differential equation (26), p. 130, i.e.

$$\frac{\partial}{\partial x}\left(h^3\frac{\partial p}{\partial x}\right) + \frac{\partial}{\partial y}\left(h^3\frac{\partial p}{\partial y}\right) + 6\mu\left(u_1\frac{\partial h}{\partial x} + v_1\frac{\partial h}{\partial y}\right) = 0;$$

$$\text{or, since} \quad h = cx, \text{ and } \frac{\partial h}{\partial y} = 0,$$

$$\frac{\partial^2 p}{\partial x^2} + \frac{3}{x}\frac{\partial p}{\partial x} + \frac{\partial^2 p}{\partial y^2} + \frac{6\mu u_1}{c^2 x^3} = 0. \quad \ldots\ldots\ldots(41)$$

This equation may be written in the form

$$\frac{\partial^2 p}{\partial x^2} + \frac{3}{x}\frac{\partial p}{\partial x} + \frac{\partial^2 p}{\partial y^2} + \frac{6\mu u_1}{c^2 x^3}\frac{4}{\pi} \times$$
$$\left(\sin\frac{\pi y}{b} + \frac{1}{3}\sin\frac{3\pi y}{b} + \ldots + \frac{1}{m}\sin\frac{m\pi y}{b} + \ldots\right) = 0,$$

since the sum of the series in brackets, for all the values of y with which we are concerned, viz. $y = 0$ to $y = b$, is $\frac{\pi}{4}$.

To solve this differential equation so as to give p as a function of x and y, it is assumed that there is a solution of the form

$$p = \Pi + p_1 + p_3 + \ldots + p_m + \ldots \text{ ad inf.} \quad \ldots.(42)$$

$$\text{in which} \quad p_m = \frac{\xi_m \sin\frac{m\pi y}{b}}{\frac{m\pi x}{b}},$$

ξ_m being a function of x only.

The integer m can have only odd values, because $p - \Pi$ must be symmetrical on both sides of $y = \frac{b}{2}$.

$$\text{Thus} \quad p - \Pi = \Sigma_1^\infty \frac{\xi_m \sin\frac{m\pi y}{b}}{\frac{m\pi x}{b}},$$

where m is odd.

If for brevity we write $24\frac{\mu u_1}{bc^2} = k$, and $\frac{m\pi x}{b} = \zeta$,

$$\frac{\partial p}{\partial x} = \frac{m\pi}{b}\frac{\partial p}{\partial \zeta} = \frac{m\pi}{b}\Sigma\left\{\frac{1}{\zeta}\frac{\partial \xi_m}{\partial \zeta} - \frac{\xi_m}{\zeta^2}\right\}\sin\frac{m\pi y}{b},$$

$$\frac{\partial^2 p}{\partial x^2} = \frac{m^2\pi^2}{b^2}\frac{\partial^2 p}{\partial \zeta^2} = \frac{m^2\pi^2}{b^2}\Sigma\left\{\frac{1}{\zeta}\frac{\partial^2 \xi_m}{\partial \zeta^2} - \frac{2}{\zeta^2}\frac{\partial \xi_m}{\partial \zeta} + \frac{2\xi^m}{\zeta^3}\right\}\sin\frac{m\pi y}{b},$$

$$\frac{\partial^2 p}{\partial y^2} = -\Sigma\frac{m^2\pi^2\xi_m}{b^2\zeta}\sin\frac{m\pi y}{b}.$$

Thus the coefficient of $\sin\frac{m\pi y}{b}$ in equation (41), p. 146, is

$$\frac{m^2\pi^2}{b^2\zeta}\left\{\frac{\partial^2 \xi_m}{\partial \zeta^2} + \frac{1}{\zeta}\frac{\partial \xi_m}{\partial \zeta} - \left(1 + \frac{1}{\zeta^2}\right)\xi_m - \frac{k}{\zeta^2}\right\} = 0. \quad \ldots.(43)$$

Every such coefficient must vanish, and consequently the factor within the brackets may be equated to zero, of which equation the particular integrals are the Bessel's Functions, $I_1(\zeta)$ and $K_1(\zeta)$, and the complete integral may be written in either of the forms

$$\xi_m = A_m I_1(\zeta) + B_m K_1(\zeta) - k\left(1 + \frac{\zeta^2}{3} + \frac{\zeta^4}{5 \cdot 3^2} + \ldots\right), \quad \ldots .(44)$$

$$\text{or} \quad \xi_m = A'_m I_1(\zeta) + B'_m K_1(\zeta) - k(\zeta^{-2} + 3\zeta^{-4} + 5 \cdot 3^2 \zeta^{-6} \ldots) \ldots (45)$$

The second form, useful when ζ is very large, being "asymptotic".

The coefficients A_m, A'_m, B_m, B'_m are to be determined so as to make ξ_m vanish for $x = a_1$, and $x = a_2$, and hence p_m vanish for all values of y on these two lines.

These coefficients can only be determined arithmetically, numerical values being given to the quantities a_1, a_2, and b. The steps of the calculation, with tables, are given in the paper (9, p. 158).

The coefficients A_m, &c., having been calculated, the values of p for as many points x, y, as may be desired are also calculated arithmetically, and when p is known the total fluid pressure supporting the block is determined by arithmetical or graphical summation, from the relation

$$P = \int_{a_1}^{a_2}\int_0^b p\,dx\,dy. \quad \ldots\ldots\ldots\ldots\ldots\ldots (46)$$

The frictional traction by (33*a*), p. 133, is $F = \frac{\mu u_1}{c} \log_e \frac{a_2}{a_1}$, per unit width, and

$$Fb = \frac{\mu u_1}{c} b \log_e \frac{a_2}{a_1} = \frac{\mu u_1}{c}\sqrt{A} \log_e \frac{a_2}{a_1} \quad \ldots\ldots (47)$$

for the whole of the square shoe, of area $A = b(a_2 - a_1)$.

The point of action of the resultant pressure is found by an arithmetical summation of moments. By way of examples, a few numerical formulæ will be given.

The total pressure on a square bearing in which

$$a_2 - a_1 = a_1 = b \text{ is } P = \frac{0{\cdot}0669 \mu u_1 A^{\frac{3}{2}}}{c^2},$$

being, by comparison with formula (31), p. 132, only 0·421 of the total pressure on a portion of equal area, equal length and inclination of a plane of infinite width, thus showing the effect of the escape of oil from the sides of the bearing.

The position of the centre of pressure for the finite square block is at a distance $0{\cdot}42a_1$ from the rear edge, as compared with $0{\cdot}431a_1$, in the infinite bearing. (See Table III, p. 136.)

The coefficient of friction $\frac{F}{P}$ is $10{\cdot}3c$. A further calculation serves to show that of the total quantity of oil which enters the interspace at the leading edge of the square shoe approximately one-sixth passes out at each of the sides and the remaining two-thirds at the rear edge.

Similarly, in the case of a bearing whose width transverse to the motion is only one-third of its length, so that

$$a_2 - a_1 = a_1 = 3b,$$

the total pressure is

$$P = \frac{0{\cdot}0146\mu u_1 a_1}{c^2} = \frac{0{\cdot}0253\mu u_1 A^{\frac{1}{2}}}{c^2},$$

and the centre of pressure is $0{\cdot}39a_1$, from the rear end.

These results, as already explained, are equally applicable to journal bearings as to plane slide bearings provided that the form of the bearing surface and the position of the pivot are such that $h = cx$, and $a_2 - a_1 = a_1$.

Arithmetical evaluations of the pressures and frictional coefficients given by the above theory have been calculated for an extensive series of bearing blocks of varying proportions by Torao Kobayashi (30).

Cylindrical Bearings of Finite Width

Mathematical treatment of cylindrical bearings of finite width, corresponding to the theory given above for plane bearings, does not yet exist. This is unfortunate, since the limitation of width has an even greater effect in a cylindrical than in a plane bearing in reducing the pressures generated, and particularly so if the arc subtended by the cylindrical bearing approaches a semicircle as it usually does in the conventional type of journal bearing.

By way of illustration of this statement, we may take a journal bearing in which the bearing-shell subtends an arc of 120°, and in which the thickness of the film at the inlet is double its thickness at the outlet (as in the plane bearing previously discussed).

Such a bearing is illustrated in fig. 24,* in which r is the radius

* From (28), p 159.

of the journal, $r + \delta$ that of the bearing-shell, the distance between their centres being ϵ, while $2b$ is the width of the bearing.

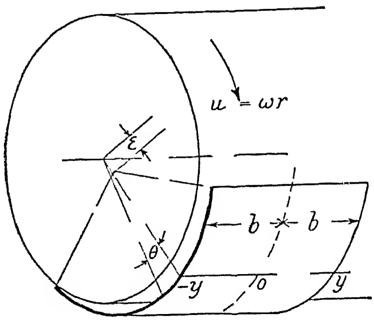


Fig. 24

It is readily seen that in such a bearing the areas through which oil may escape at the sides of the bearing are much greater relatively to the areas at the front and rear (at which the oil would enter and leave in a two-dimensional bearing) than is the case in a plane bearing of similar length and width.

In the particular case in which the width of the bearing is equal to the radius of the journal, and in which the radii of journal and bearing are equal, so that

$$\delta = 0$$
$$2b = r,$$

the respective areas of the leading, trailing and side openings are in the proportions

$$2 : 1 : 10{\cdot}3.$$

In other words, the oil which enters the interspace at the leading edge has more than 10 times greater area by which to leave at the sides than at the rear. Pressures are consequently determined almost entirely by the conditions at the sides, and a two-dimensional solution would convey a very false idea of the actual conditions.

A more useful approximation in such a case can be obtained by treating the bearing as infinitely long in comparison with its width. On this assumption the pressures over the portion of the bearing near its trailing end (which is the only portion in which effective pressures will be generated) are given by

$$p = \frac{3\mu\omega\epsilon \sin\theta}{h^3}(b^2 - y^2),$$

y being the axial co-ordinate measured from the middle circumference, and h the varying thickness, determined by

$$h = \delta + \epsilon \cos\theta.$$

Experimental Results

The curves given in the right-hand half of fig. 24*a* are derived from an extensive series of tests of a pivoted journal bearing of which the circumferential length was 6·98 cm. and the width was 6·35 cm., the block being thus not quite square. From examination of this diagram it will be found that for a given load the coefficient of friction varies approximately as $\sqrt{\mu u_1}$, while for a given value of μu_1, it varies nearly inversely as the square root of the load, both these results being in accordance with the formulæ above. The facts stated are brought out more explicitly in the following table, which shows that the values of F, P and μu_1, as read off the right-hand part of fig. 24*a*, make $F\sqrt{(P/\mu u_1)}$ approximately constant. The left-hand part of fig. 24*a* will be explained on p. 157.

TABLE V

F.	$P/\mu u_1$.	$\sqrt{(P/\mu u_1)}$.	$10^3 F\sqrt{(P/\mu u_1)}$.
0·0008	0·12	0·35	0·28
·0012	·067	·26	·31
·0016	·037	·19	·30
·0020	·023	·15	·30
·0024	·017	·13	·31

Types of Pivoted Bearings

The chief practical field of application of *plane* pivoted bearings is to thrust bearings. These usually take the form of an annular series of " shoes " or " blocks " pivoted upon fixed points in the stationary casing and presenting their plane working surfaces to a plane-surfaced annular collar fixed on the rotating shaft. Such a

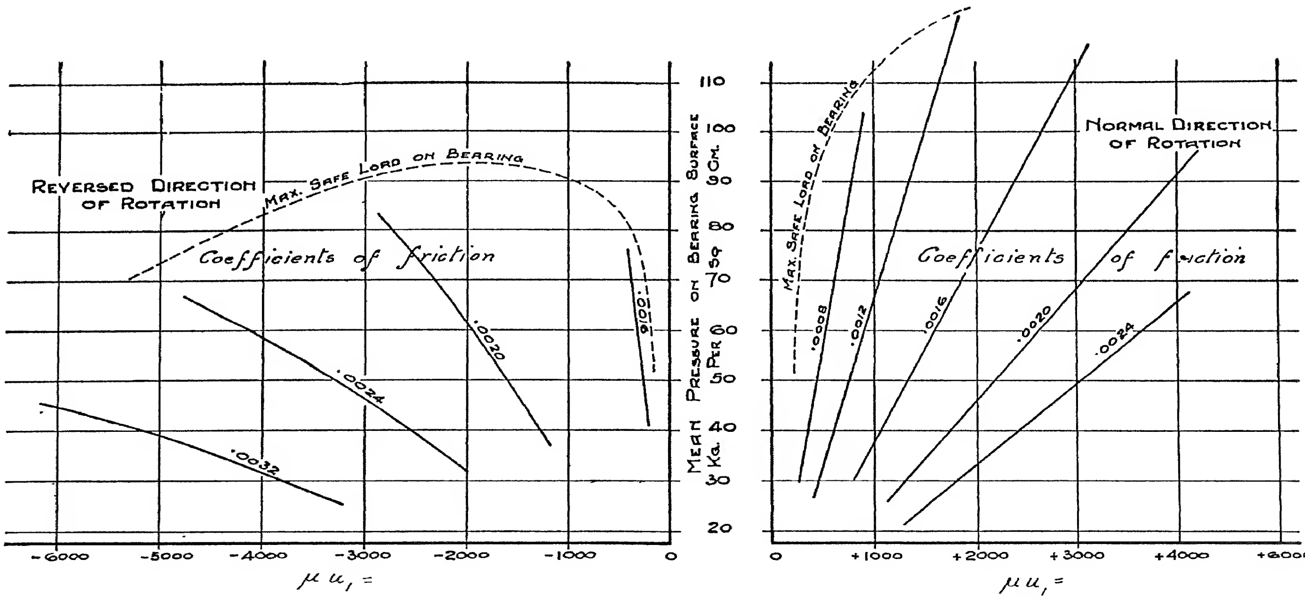


Fig. 24a.—The base is the product of the viscosity, μ, of the oil at air temperature, and the surface speed of the journal (positive or negative), both in C G.S. units

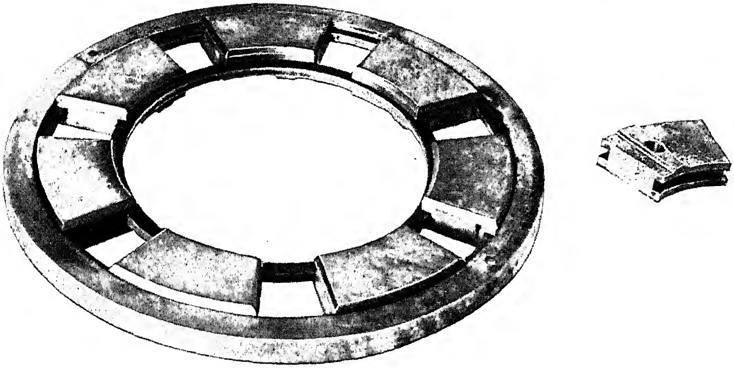


FIG. 26.—THRUST SHOES

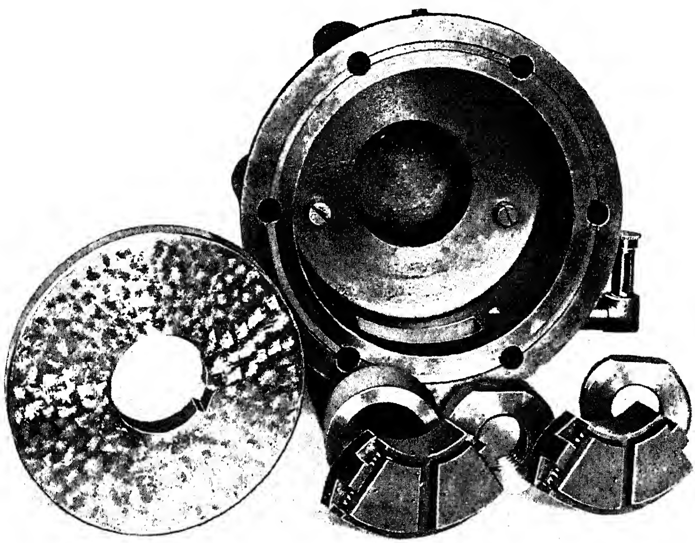


FIG. 28.—THRUST BEARING FOR HORIZONTAL SHAFTS

thrust bearing arranged for a vertical shaft is shown in fig. 25. In this bearing the thrust shoes, *t*, are fixed in the lower part of the casing of the bearing which also serves as the casing of a journal bearing for the thrust shaft. (The journal bearing is of the flexibly-pivoted type described on p. 155.)

In order that the casing may form a reservoir for oil to

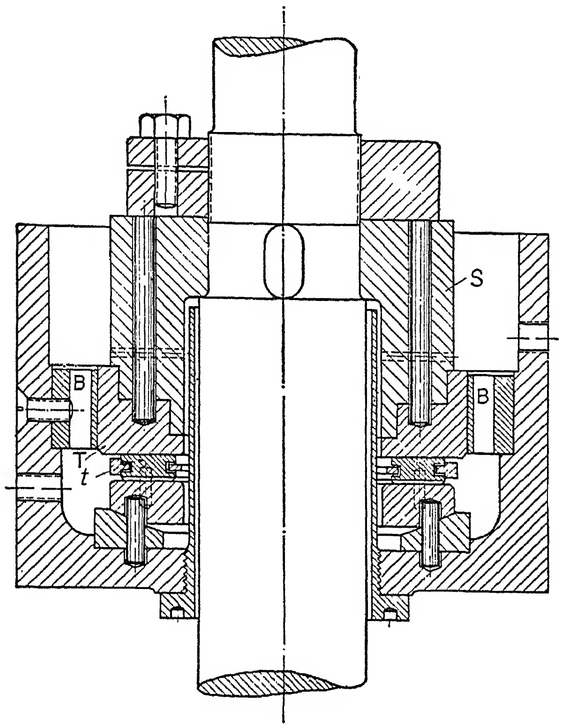


Fig. 25

lubricate both bearings, the journal surface is formed not directly upon the shaft, but on the outer surface of a collar T, attached to the sleeve S, and forming also the thrust collar which revolves upon the annulus of thrust shoes *t*. This annulus is shown separately in fig. 26. (The flexible journal ring is shown in fig. 31, facing p. 154).

In figs. 27 and 28 is shown another type of thrust bearing, convenient for application to horizontal shafts. In this form, which is adapted to take thrusts in either axial direction, two pivoted thrust-shoes R only are employed for each direction of thrust, each pair being mounted in a common housing H, which is itself pivoted on the

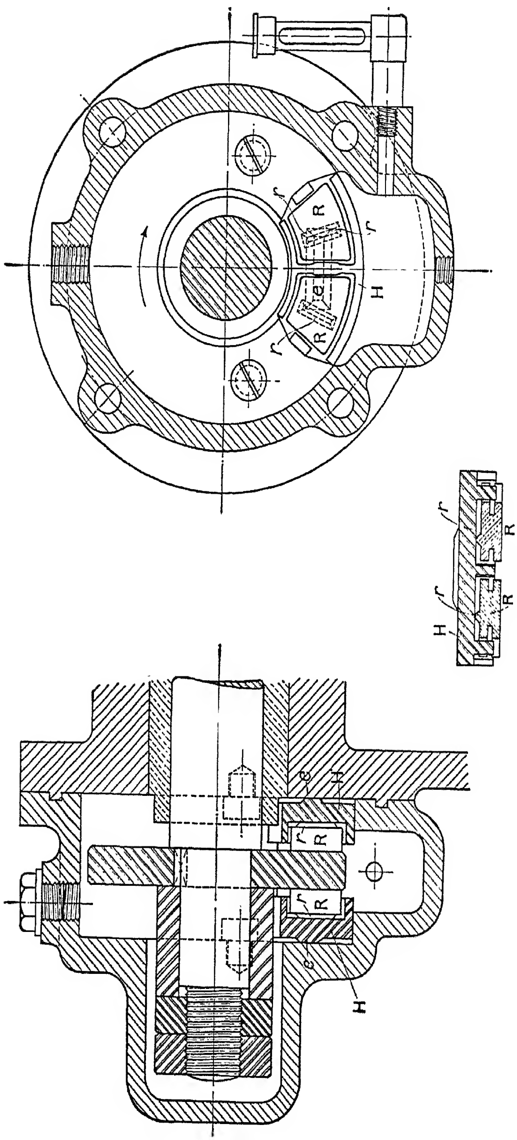


Fig. 27

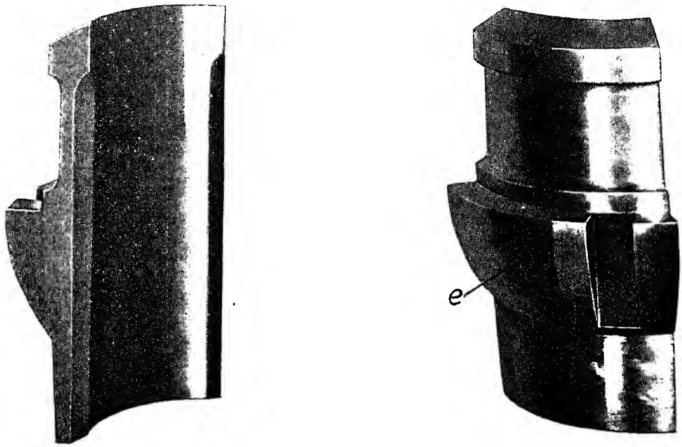


FIG. 29.—VIEWS OF PIVOTED JOURNAL BEARING SHOE

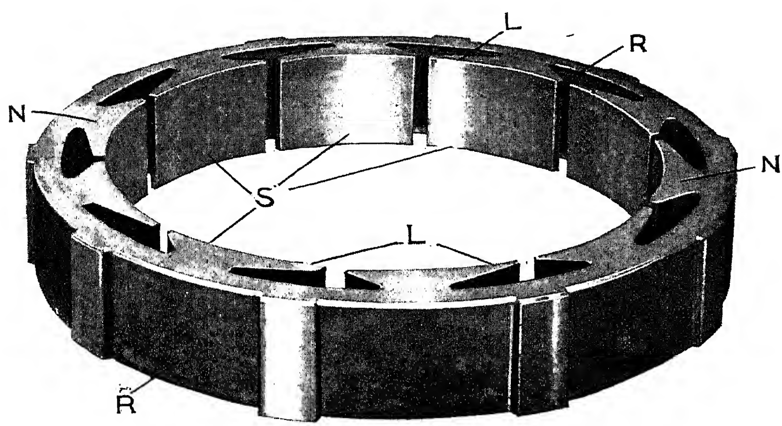


FIG. 31.—LARGE JOURNAL BEARING

lower part of the fixed casing on an edge *e* at right angles to the pivoting ribs, *rr*, of the individual shoes.

Fig. 28 is a photograph of the parts of the bearing which is shown by longitudinal and cross sections in fig. 27.

In fig. 29 are shown two views of a pivoted journal bearing shoe, being one of an annular series of four arranged for the journal bearing of a vertical shaft. The pivoting edge *e* is clearly seen on the back of the shoe.

Flexible Bearings

The comparatively small clearances and slight relative inclinations between coacting bearing parts, requisite to produce effective lubricating films, allow of a modified type of construction for achieving the same purposes as are attained by pivoted bearings. It is evident that in these a spring, or other continuous but deformable connection, may be substituted for the rolling or rotating contact of a pivot. Such a spring may be either a separate part attached both to the shoe and its supporting member, or may be an integral part of one or both of these provided such part is made with the necessary degree of flexibility to allow of the shoe deflecting under the load. Alternatively, as in a type of construction proposed by Ferranti,* a pair of springs may be used to connect the shoe and its support, viz. a comparatively stiff spring at the rear and a lighter spring at the front of the shoe. This construction, which is illustrated in fig. 30, will evidently have the effect of applying the resultant load at a point P, behind the middle point of the shoe, much as if it were applied to a rigid pivot at that point.

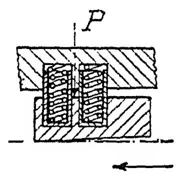


Fig. 30

The chief advantage of a flexible construction is that it enables small or relatively unimportant bearings to be simplified, by constructing a number of bearing shoes integral with, but flexibly connected to, a common supporting member. A serious disadvantage is that the flexibility involves more or less risk of fracture of the flexible part, a danger which is to some extent overcome by giving flexibility to a portion of the shoe itself. The large journal bearing, already mentioned on p. 153, is constructed in this way, and is illustrated in fig. 31, and in fig. 25, p. 153.

In the former the individual shoes, S, may be seen attached to the

* British patent No. 5035/1910.

supporting ring R, by flexible necks N, and having also their leading portions, L, reduced in thickness for some distance from the leading edges.

Limitations of the Theory

As the shoes of such thrust bearings as are illustrated in figs. 24*a* to 28 are usually of small radial width compared to their mean radii, the formulæ given for rectangular bearing slippers may usually be applied to them with sufficient accuracy for practical purposes in spite of their sectorial form. A more exact calculation can be made when required by a process which refers the co-ordinates of the sectorial shoe to those of the rectangular shoe.*

Of greater practical importance are the departures from the results of the calculations which in some cases arise from the insufficiency of the physical assumptions which have been made, especially as to the constancy of the coefficient of viscosity.

An experimental method of solution, imagined and applied by Kingsbury (29), is free from most, if not all, of these limitations. This method utilizes the identity which exists between the equations connecting pressure and volume-flow in viscous liquids, and potential and current in an electrical conductor. The conductor used is a conducting liquid contained within solid, non-conducting boundaries shaped to represent in correct proportion (though on an exaggerated scale as regards thickness of the conductor) the lubricating film to be investigated. The results obtained by Professor Kingsbury agree closely with those of the mathematical investigations, e.g. those of the plane bearing of finite width given on pp. 146 of the present chapter. The method has been applied to both plane and cylindrical bearings of various ratios of length to width.

It was shown in Table I, p. 118, that the viscosity of lubricating oils diminishes rapidly as the temperature rises. In a well-loaded pivoted bearing, carrying for instance a mean pressure of 70 Kgm. per square centimetre, and with the product μu_1 amounting to 2000 C.G.S., and with usual dimensions, it can easily be deduced from calculations of the energy expended in overcoming the viscous friction, and of the heat capacity of the quantity of oil flowing through the lubricating film, that apart from conduction of heat through the metal, the oil would rise in temperature some 50° C. in passing

* See (16, p. 158) Correspondence.

through the bearing. Conduction will diminish this rise of temperature, but in most cases of heavily loaded bearings it is still sufficient to make the viscosity of the oil in the rear portion of the film much lower than in the leading portion. Thus, other conditions remaining unaltered, the outflow of oil at the rear will take place with a less rapid fall of pressure in that direction, and the point of maximum pressure will be shifted towards the front of the bearing. In fig. 18 the dotted curve $p\frac{1}{2}$, p. 135, *Revue B.B.C.* (19, p. 159), is figured on the assumption that the rise of temperature of the oil is such that its viscosity at exit is reduced to one-half of its value at entry, the conditions being otherwise the same as those for the full-line curve as already explained on p. 135.

The lower values of the fluid pressure throughout the film and the shift of the point of maximum pressure towards the leading edge are clearly seen. The point of action of the resultant pressure is also moved forward relatively to its position with constant oil temperature, and it may even happen that the centre of pressure is at, or in front of, the middle point of the bearing block. If, for example, the direction of motion of a pivoted bearing is reversed, so that the pivot is before instead of behind the centre of the bearing, it is still possible in many cases for a lubricating film to be formed and pressures generated in it in equilibrium with the load. Such an effect is shown in the left-hand half of fig. 24*a*, which shows the results of reversing the bearing. In such a case the oil film is necessarily thinner, and the coefficient of friction higher than for the correct direction of motion, but nevertheless the capability of being reversed in this manner, and of even then working with coefficients of friction lower than those of non-pivoted bearings, is a valuable property of the pivoted type. When, however, pivoted bearings are employed in this manner, it has always to be remembered that their success when running reversed depends upon the lubricant having a considerable rate of diminution of viscosity with rising temperature. For example, an experimental thrust bearing which ran very successfully in both directions with water and with a mineral oil of low viscosity as lubricants, or with carbon bisulphide when running in the normal direction, completely failed to run in the reversed direction with the last-named fluid, doubtless on account of the peculiarity of its viscosity-temperature relation, which has already been mentioned on p. 117.

Effects of the same nature, which arise in the use of air as a lubricant in pivoted thrust bearings, have been pointed out and

experimentally investigated by Stone * (24, p. 159). With air, owing to the viscosity of gases increasing with rising temperatures instead of diminishing as in liquids, pivoted bearings tend to be much less stable as to the inclination of the pivoted shoe than with liquid lubricants.

On the other hand, as the same author has also remarked, the increase of the viscosity of the air film with temperature tends to increase the thickness of the film when a rise of temperature takes place owing to excessive load or undue resistance. The risk of direct contact of the bearing elements thus tends to become less as the bearing heats up, instead of greater as with liquid lubricants.

Calculation and experiment agree in showing that the successful use of air as a lubricant demands the highest refinements of workmanship, with moderate loads and relatively high speeds.

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* These experiments were made by means of a thrust bearing consisting of quartz-crystal thrust shoes and a glass thrust collar, the bearing surfaces being worked to true planes by optical methods. Monochromatic diffraction bands produced by the closely adjacent pair of bearing surfaces at a slight mutual inclination gave an immediate and very accurate measure of the thickness of the lubricating film.

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CHAPTER IV

Stream-line and Turbulent Flow

Stream-line Motion

The motion of a fluid may be conveniently studied by considering the distribution and history of the *stream lines*, i.e. the actual paths of the particles. If these paths or stream lines preserve their configuration unchanged, the motion is called *steady* or *stream-line* motion. (See Chapter II, p. 57.)

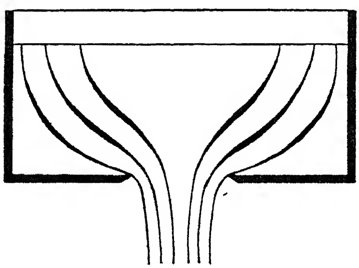

Fig. 1

If the stream line be imagined to form the axis of a tube of finite sectional area having imaginary boundaries, and such that its area at different points in its length is inversely proportional to the velocities at these points, this is termed a "stream tube".*

Such stream lines must always have a continuous curvature, since, to cause a sudden change in direction, an infinite force acting at right angles to the direction of flow would be necessary. It follows that in steady motion a fluid will always move in a curve around any sharp corner, and that the stream lines will always be tangential to such boundaries, as indicated in fig. 1, which shows the general form of the stream lines of flow from a sharp-edged orifice. With a very viscous fluid, the effect of cohesion may introduce comparatively large forces, and the radius or curvature may then become very small.

* See an alternative way of putting this idea, Chapter II, p. 57.

Stability of Stream-line Motion

Several conditions combine to determine whether, in any particular case of flow, the motion of a fluid shall be stream-line or turbulent. Osborne Reynolds, who first investigated the two manners of motion by the method of colour bands,* came to the conclusion that the conditions tending to the maintenance of stream-line motion are:

(1) an increase in the viscosity of the fluid;
(2) converging solid boundaries;
(3) free (exposed to air) surfaces;
(4) curvature of the path with the greatest velocity at the outside of the curve;
(5) a reduced density of the fluid.

The reverse of these conditions tends to give rise to turbulence, as does a state of affairs in which a stream of fluid is projected into a body of fluid at rest.

The effect of solid boundaries in producing turbulence would appear to be due rather to their tangential than to their lateral stiffness. One remarkable instance of this effect of a boundary possessing tangential stiffness is shown by the effect of a film of oil on the surface of water exposed to the wind. The oil film exerts a very small but appreciable tangential constraint, with the result that the motion of the water below the film tends to become unstable. This results in the formation of eddies below the surface, and the energy, which is otherwise imparted by the action of the wind to form and maintain stable wave motion, is now absorbed in the institution of eddy motion, with the well-known effect as to the stilling of the waves.

Where two streams of fluid are moving with different velocities the common surface of separation is in a very unstable condition. Reynolds showed this by allowing the two liquids, carbon bisulphide and water, to form a horizontal surface of separation in a long horizontal tube. The tube was then slightly tilted so as to produce a relative axial motion of the fluids, when it was found that the motion was unstable for extremely small values of the relative velocity.

This also explains why diverging boundaries are such a cause of turbulence. Experiment shows that in such a case as shown in

* *Phil. Trans. Roy. Soc.*, 1883.

fig. 2 the high-velocity fluid leaving the pipe of small section is projected as a core into the surrounding mass of dead water, thereby giving rise to the conditions necessary for eddy formation.

More recent experiments* tend to show that the foregoing conclusions as to the effect of the curvature of the paths in affecting the manner of motion, are only true where the outer boundary of

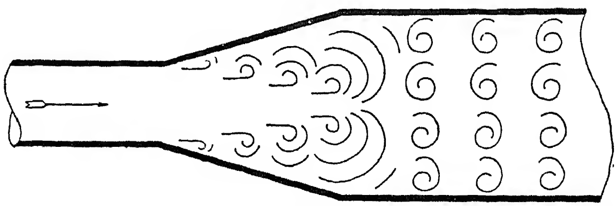

Fig. 2

the fluid is formed by a solid surface, and that in some cases—as shown at the impact of a steady jet on a plane surface, at the efflux of a jet from a sharp-edged orifice and in motion in a free vortex—curved motion, with the velocity greatest at the inside and not at the outside of the curve, tends to stream-line motion. Generally speaking, wherever the velocity of flow is increasing and the pressure diminishing, as where lines of flow are converging, there is an overwhelming tendency to stability of flow. In a tube with converging boundaries it is this which leads to stability, and it is because this effect is sufficiently great to overcome the tendency to turbulent motion to which all solid boundaries, of whatever form, give rise, that the motion in such tubes is stable for very high velocities.

Hele Shaw's Experiments

The fact that stream-line motion is possible at fairly high velocities between parallel boundaries if the fluid is viscous, and if the distance between the boundaries is small, has been taken advantage of by Dr. Hele Shaw,† who produced stream-line motion in the flow of glycerine between two parallel glass plates, and showed the form of the stream lines by introducing coloured dye solution at a number of points. By inserting obstacles between the glass plates the form of the stream lines corresponding to flow

* *Memoirs, Manchester Lit. and Phil. Soc.*, 55, 1911, No. 13.

† *Trans. Inst. Naval Architects*, 1898, p. 27.

through a passage or around a body of any required shape can thus be obtained (figs. 3 and 4).

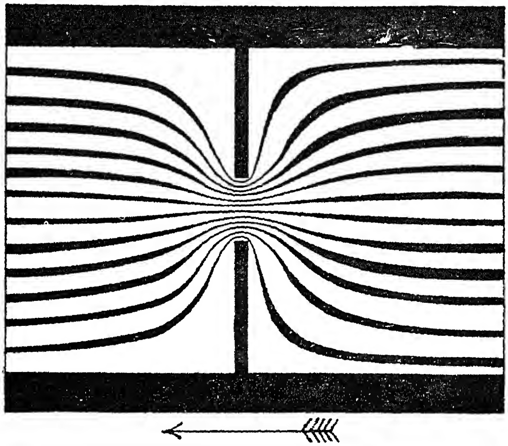

Fig. 3

The form of the lines to be expected in the case of two-dimensional flow of a perfect non-viscous fluid around bodies of simple and

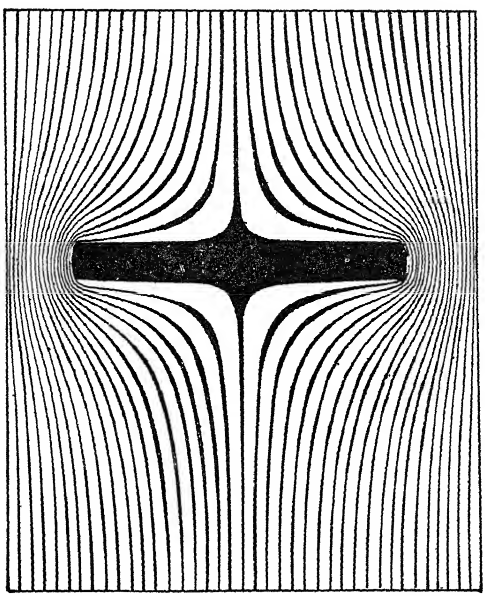

Fig. 4

symmetrical shape, may be calculated,* and an examination of the stream lines obtained in the Hele Shaw apparatus shows that they are identical in form with those thus obtained by calculation, in spite of the fact that in one case the forces operating are entirely due to inertia, and in the other to viscosity. It has been shown by Sir George Stokes† that this is to be expected, for if PQ and PQ′ (fig. 5) be two boundaries of a stream tube, and if PP′ and QQ′ be normals to one of the boundaries, ultimately these will become elements of two consecutive equipotential lines, and if produced will meet on the centre of curvature of the tube, so that if v and $v + \delta v$ be the velocities at P′ and P, and if r be the curvature and t the thickness,

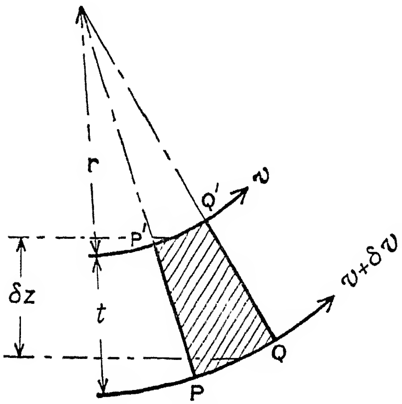

Fig. 5

$$\frac{v + \delta v}{v} = \frac{\mathrm{PQ}}{\mathrm{P'Q'}} = \frac{r + t}{r};$$

$$\therefore \frac{\delta v}{t} = \frac{v}{r} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Again considering the equilibrium of the element P′Q′QP, now imagined as part of a perfect non-viscous fluid, the centrifugal force will be balanced by the difference of normal pressures (δp) on the inner and outer faces, and by the resolved part of the difference of pressure due to the difference of level (δz) between the two faces. If directions towards the centre of curvature be called positive, on resolving normally,

$$\frac{\rho v^2 t}{r} = -\delta p - \rho \delta z \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

On substituting for $\frac{vt}{r}$ from (1) this becomes

$$\rho v \delta v + \delta p + \rho \delta z = 0$$

$$\text{or} \qquad \frac{\rho v^2}{2} + p + \rho z = \text{constant},$$

* *Hydrodynamics*, Lamb, p. 61; also *Trans. Inst. N. A.*, 1898.
† *British Association Reports*, 1898, pp. 143–4.

which is Bernoulli's equation of energy for a perfect non-viscous fluid. It follows that the velocity relationship indicated in (1), p. 164, which obtains when viscosity is the dominating factor, is also consistent with the stream-line flow of a non-viscous fluid.

Critical Velocity

The nature of the two modes of fluid motion was first demonstrated by Osborne Reynolds* in a series of experiments on parallel glass tubes of various diameters up to 2 in. These were fitted with bell-mouthed entrances and were immersed horizontally in a tank

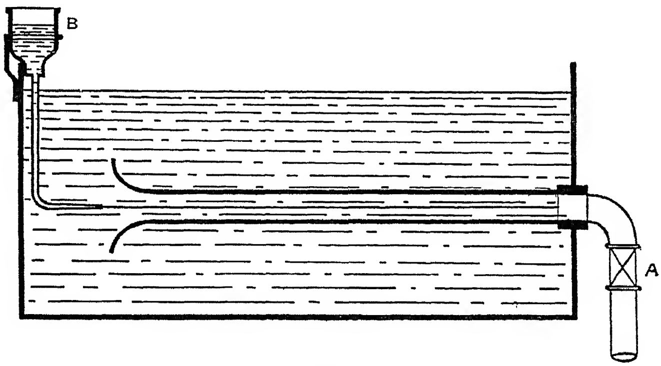

Fig. 6

of water having glass sides (fig. 6). In these experiments the water in the tank was allowed to stand until motionless. The outlet valve A was then opened, allowing water to flow slowly through the tube. A little water coloured with aniline dye was introduced at the entrance to the tube through a fine tube supplied from the vessel B.

At low velocities this fluid is drawn out into a single colour band extending through the length of the tube. This appears to be motionless unless a slight movement of oscillation is given to the water in the supply tank, when the colour band sways from side to side, but without losing its definition. As the velocity of flow is gradually increased, by opening the outlet valve, the colour band becomes more attenuated, still, however, retaining its definition, until at a certain velocity eddies begin to be formed, at first intermittently, near the outlet end of the tube (fig. 7). As the velocity is still further increased the point of eddy initiation approaches the

* *Phil. Trans. Roy. Soc.*, 1883.

mouthpiece, and finally the motion becomes sinuous throughout. The apparent lesser tendency to eddy formation near the inlet end of the tube is due to the stabilizing influence of the convergent mouthpiece.

The velocity at which eddy formation is first noted in a long

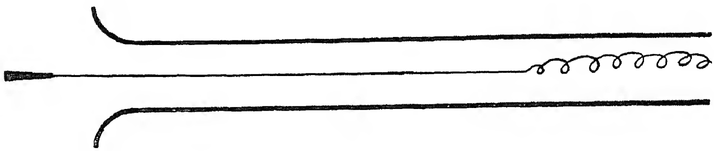

Fig. 7

tube in such experiments is termed the "higher critical velocity". There is also a "lower critical velocity", at which the eddies in originally turbulent flow die out, and this is, strictly speaking, the true critical velocity. It has a much more definite value than the higher critical velocity, which is extremely sensitive to any disturbance, either of the fluid before entering the tube, or at the entrance. Over the range of velocities between the two critical values, the

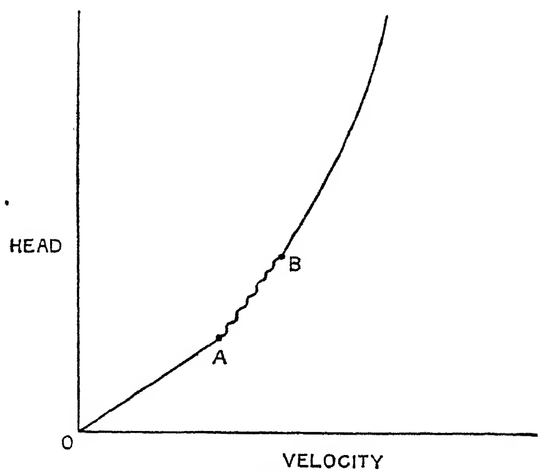

Fig. 8

fluid, if moving with stream-line flow, is in an essentially unstable state, and the slightest disturbance may cause it to break down into turbulent motion.

The determination of the lower critical velocity is not possible by the colour-band method, and Reynolds took advantage of the

fact that the law of resistance changes at the critical velocity, to determine the values by measuring the loss of head accompanying different velocities of flow in pipes of different diameters. On plotting a curve showing velocities and losses of head (fig. 8) it is found that up to a certain velocity, A, for any pipe, the points lie on a straight line passing through the origin of co-ordinates. From A to B there is a range of velocities over which the plotted points are very irregular, indicating general instability, while for greater velocities the points lie on a smooth curve, indicating that the loss of head is possibly proportional to v^n.

To test this, and if so to determine the value of n, the logarithms

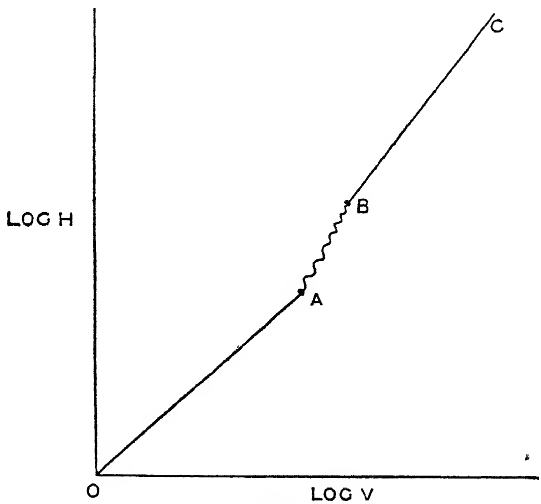

Fig. 9

of the loss of head h and of the velocity were plotted (fig. 9). Then if

$$h = k\,v^n,$$
$$\log h = \log k + n \log v,$$

the equation to a straight line inclined at an angle $\tan^{-1} n$ to the axis of $\log v$, and cutting off an intercept $\log k$ on the axis of $\log h$.

On doing this it is found that if the velocity is initially turbulent the plotted points lie on a straight line up to a certain point A, the value of n for this portion of the range being unity. At A, which marks the lower critical velocity, the law suddenly changes and h increases rapidly. There is, however, no definite relationship between h and v until the point B is reached. Above

this point the relationship again becomes definite, and within the limits of experimental error, over a moderate range of velocities, the plotted points lie on a straight line whose inclination varies with the roughness of the pipe walls. The values of n determined in this way by Reynolds are:

Material of Pipe.	n.
Lead	1·79
Varnished	1·82
Glass	1·79
New Cast Iron ..	1·88
Old Cast Iron ..	2·0

those for cast iron being deduced from experiments by Darcy.

When tested over a wide range of velocities, it is found that the value of n in the case of a smooth-walled pipe is not constant but increases somewhat as the velocity is increased.

Between A and B the value of n is greater than between B and C, and the increased resistance accompanying a given change in velocity is greater even than when the motion is entirely turbulent. This is due to the fact that within this range of velocities eddies are being initiated in the tube, and the loss of head is due not only to the maintenance of a more or less uniform eddy regime, but also to the initiation of eddy motion.

Messrs. Barnes and Coker* have determined the critical velocity in pipe flow by allowing water to flow through the given pipe which was jacketed with water at a higher temperature. The temperature of the water discharging from the pipe was measured by a delicate thermometer. So long as the motion is non-sinuous, transmission of heat through the water is entirely due to conduction and is extremely slow, so that the thermometer gives a steady reading sensibly the same as that in the supply tank. Immediately the critical velocity is attained, the rate of heat transmission is increased due to convection, and the change from stream-line to turbulent motion is marked by a sudden increase in the temperature of the discharge.

* *Proc. Roy. Soc.* A, 74.

The law governing the relationship between the critical velocity and the factors involved was deduced by Reynolds from a consideration of the equations of motion: for if the state of motion be supposed to depend on the mean velocity in the tube and on the diameter, the acceleration may be expressed as the difference of two terms, one of which is of the nature $\mu v/d$, and the other of the nature ρv^2. It was then inferred that since the relative value of these terms probably determines the critical velocity, the latter will depend on some particular value of the ratio $\mu/\rho v d$. To test the accuracy of this conclusion experiments were made on pipes of different diameters, and with different values of μ obtained by varying the temperature of the water between 5° C. and 22° C.

The results of the experiments fully justified the foregoing conclusions, and showed that the critical velocity in a straight parallel pipe is given by the formula

$$v_k = \frac{P}{bd},$$

where b is a numerical constant, and where $P \propto \mu/\rho$. If the unit of length is the foot, b equals 25·8 for the lower critical velocity, and 4·06 for the higher velocity; while if t = temperature in degrees Centigrade,

$$P = \frac{1}{1 + 0{\cdot}03368t + 0{\cdot}000221t^2}.$$

More recent experiments by Coker, Clement, and Barnes* and others carried out by Ekman† on the original apparatus of Reynolds, show that by taking the greatest care to eliminate all disturbance at entry to the tube, values of the higher critical velocity considerably greater than (up to 3·66 times as great as) those given by the above formula may be obtained. The probability is, in fact, that there is no definite *higher* critical velocity, but that this always increases with decreasing disturbances.

A general expression for the lower critical velocity in a parallel pipe, applicable to any fluid and any system of units, is

$$v_k = \frac{2300\mu}{d\rho} \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

$$= \frac{2300\nu}{d}.$$

* *Trans. Roy. Soc.*, 1903; *Proc. Roy. Soc.* A, 74.
† "Arkiv för Matematic", *Ast. Och. Fys.*, 1910, 6, No. 12.

Thus for water at 0° C., $\mu/\rho = \nu = 1{\cdot}92 \times 10^{-5}$ in foot-pound second units, so that

$$v_k = \frac{0{\cdot}0442}{d} \text{ ft.-sec., where } d \text{ is in feet.}$$

While for air at 0° C.,

$$\nu = 14{\cdot}15 \times 10^{-5};$$

$$\therefore\ v_k = \frac{0{\cdot}326}{d} \text{ ft.-sec. where } d \text{ is in feet.}$$

In this connection fig. 10* is of interest, as showing the results of experiments on a number of pipes of different diameters, with air and water flow, in which values of $R/\rho v^2$ are plotted as ordinates against the corresponding values of vd/ν or of $\log vd/\nu$. Here R is the surface friction per unit area of the pipe. The curve consists of two parts connected by a narrow vertical band corresponding to a value of vd/ν of approximately 2300, over which the points for the various pipes are somewhat irregularly disposed. This band indicates the range of instability between stream line and true turbulent flow. The left-hand curve, corresponding to speeds below the critical value, is calculated from the formula

$$R = \frac{8\mu v}{d},$$

theoretically corresponding to stream-line flow.† It will be seen that the points for both air and water flow lie closely on this curve, and that the break-down of the stream-line motion takes place in all cases at approximately the same value of vd/ν.

As may be shown by an application of the principle of dynamical similarity,‡ formula (3) is a particular case of the general formula

$$v_k = \frac{k\nu}{l},$$

which is applicable to all cases of fluid motion. Here l is the length of some one definite dimension of the body. The value of the constant k now depends only on the form of the surfaces over which flow is taking place. Thus in flow past similar plates immersed in water and in air, Eden § has shown by visual observation that the

* Stanton and Pannell, *Phil. Trans. Roy. Soc.* A, 214.

† Chap. V. p. 192. ‡ Chap. V, p. 185.

§ *Advisory Committee for Aeronautics*, T.R., 1910–11, p. 48.

type of flow, especially in the rear of the plate, is identical for identical values of vl/ν, where l is the length of any particular side of the plate.

Critical Velocity in Converging Tubes

In a converging tube the angle of convergence of the sides has a large effect on the critical velocity. At all ordinary velocities the motion in tubes or nozzles having more than a few degrees of convergence may be considered as non-sinuous. Experiments on the flow of water through circular pipes having sides converging uniformly at an angle θ gave the following approximate values for the lower critical velocity, at 14° C.*

θ.		5 Deg.	7·5 Deg.	10 Deg.	15 Deg.
Critical Velocity, ft.-sec.	At large section (3 in. diameter)	1·5	1·94	2·44	3·25
	At throat ($1\frac{1}{2}$ in. diameter)	6·0	7·76	9·77	12·9
	At mean section ($2\frac{1}{4}$ in. diameter)	2·7	3·45	4·34	5·73

The lower critical velocity in a $1\frac{1}{2}$-in. parallel pipe at this temperature is 0·20 ft. per second. Should the ratio of higher to lower critical velocities have the same value in a conical pipe as in a parallel pipe, this would mean that in the case of a $1\frac{1}{2}$-in. jet discharging from a converging nozzle with steady flow in the supply pipe, the critical velocity would have the following values.

θ	5 Degrees.	7·5 Degrees.	10 Degrees.	15 Degrees.
Critical velocity, ft.-sec.	39	50	63	84

In flow through a pipe bend, the velocity at which the resistance ceases to obey the laws of laminar flow is less than in a straight pipe. There is now a considerable range of velocity over which the resistance is proportional to a power of the velocity higher than unity, but in which turbulence is not developed. This is due to the develop-

* Gibson, *Proc. Roy. Soc.* A, 83, 1910, p. 376.

ment of a cross circulatory current superposed on the laminar flow. The critical velocity is not well defined but experiments * indicate that full turbulence is developed at a somewhat higher velocity than in straight pipes.

The Measurement of the Velocity of Flow in Fluids

Several methods are available for measuring the flow of fluids in pipes. Of these, the use of the Venturi meter or of the Pitot tube are the most common. Recent investigations into the possibilities of the hot-wire anemometer have shown that this is capable of giving excellent results, and that it is likely to be especially valuable for the measurement of pulsating flow.

The Venturi Meter

The Venturi meter, invented by Clemens Herschel in 1881, affords perhaps the simplest means of measuring the flow of a liquid. When fitted to a pipe line of diameter greater than about 2 in. its indications are, under normal conditions, thoroughly reliable so long as the

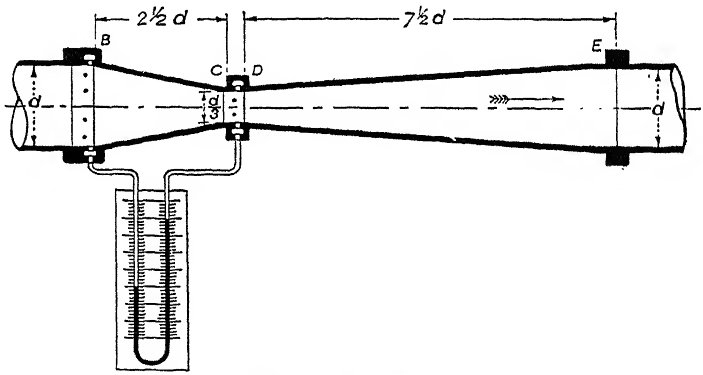

Fig. 11.—Venturi Meter.

velocity in the pipe line exceeds 1 ft. per second, and the discharge may then be predicted, even without calibration, to within 1 or 2 per cent.

The meter is usually constructed of approximately the proportions shown in fig. 11, and consists essentially of an upstream cone usually having an angle of convergence of about 20°, connected to

* C. M. White, *Proc. Roy. Soc.* A, **123**, 1929, p. 645.

a downstream cone whose angle of divergence is about 5° 30′, by easy curves. One annular chamber surrounds the entrance to the meter, and a second surrounds the throat, the mean pressures at these points being transmitted to these chambers through a series of small holes in the wall of the pipe. The two chambers are connected to the two limbs of a differential pressure gauge which records their difference of pressure h in feet of water. For this purpose a U-tube containing mercury may be used as in fig. 11. In this case if the connecting pipes are full of water it may readily be shown that the difference of pressure in feet of water is equal to 12·59 times the difference of level of the tops of the mercury columns. By using an inverted U-tube with compressed air supplied to the highest portion of the tube, the difference of pressure may be directly recorded in feet of water. When an automatic record is desired, the type of mechanism shown in fig. 12 may be used.

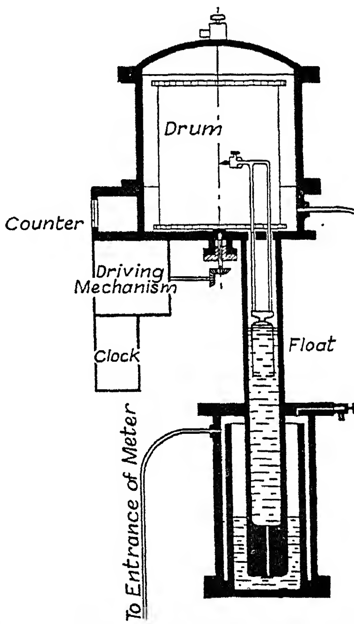

Fig. 12.—Recording Mechanism for Venturi Meter.

If P, A, V and p, a, v represent the pressures in pounds per square feet, the areas in square feet, and the mean velocities in feet per second respectively at the entrance and throat of a meter whose axis is horizontal, neglecting any loss of energy between entrance and throat, Bernoulli's equation of energy becomes

$$\frac{P}{W} + \frac{V^2}{2g} = \frac{p}{W} + \frac{V^2}{2g}\left(\frac{A}{a}\right)^2;$$

$$\therefore \frac{P - p}{W} = \frac{V^2}{2g}\left\{\left(\frac{A}{a}\right)^2 - 1\right\}$$

$$\text{or} \qquad V = \sqrt{\frac{2gh}{\left(\frac{A}{a}\right)^2 - 1}} \text{ ft.-sec.} \ldots\ldots (4)$$

Actually owing mainly to frictional losses the velocity is slightly less than is indicated by formula (4), and is given by

$$V = C\sqrt{\frac{2gh}{\left(\frac{A}{a}\right)^2 - 1}} \text{ ft.-sec.,} \quad \ldots\ldots\ldots\ldots (5)$$

where C varies from about 0·96 to 0·995,* usually increasing slightly with the size of meter. When used to measure pulsating flow, the value of C is reduced. The effect is, however, small for any such percentage fluctuations of velocity as are usual in practice, even with the discharge from a reciprocating pump. For accurate results the meter should be installed in a straight length of pipe removed from the influence of bends. Such bends set up whirling flow in the pipe, and this tends to increase the effective value of C.

The Venturi meter may also be used to measure the flow of gases.† In this case, for air, the discharge is given by

$$Q = CA\beta\sqrt{2gP_1W_1}, \quad \ldots\ldots\ldots\ldots\ldots (6)$$

where P_1 is the pressure at entrance in pounds per square foot,
W_1 is the weight per cubic foot at P and temperature T;
and where, if p_1 is the pressure at entrance in pounds per square inch,
p_2 is the pressure at throat in pounds per square inch,
m is the ratio of areas at entrance and throat,
n is the index of expansion (1·408 for dry air expanding adiabatically),

$$\beta = \sqrt{\frac{\frac{n}{n-1}\left\{1 - \left(\frac{p_2}{p_1}\right)^{\frac{n-1}{n}}\right\}\left(\frac{p_2}{p_1}\right)^{\frac{2}{n}}}{m^2 - \left(\frac{p_2}{p_1}\right)^{\frac{2}{n}}}}.$$

If τ_1 be the absolute temperature at entrance on the Fahrenheit scale, on writing $W_1 = \frac{2{\cdot}70 p_1}{\tau_1}$ (the value for dry air) (6) reduces to

$$Q = 1{\cdot}10 C a_1 \beta \frac{p_1}{\sqrt{\tau_1}} \text{ lb. per second,} \quad \ldots\ldots\ldots (7)$$

where a_1 is the area at entrance in square inches.

* For a discussion of the variability of the coefficient C, see "Abnormal Coefficients of the Venturi Meter", *Proc. Inst. C. E.*, **199**, 1914–5, Part I.

† "Measurement of Air Flow by Venturi Meter", *Proc. Inst. Mech. E.*, 1919, p. 593; "Commercial Metering of Air, Gas, and Steam", *Proc. Inst. C. E.*, 1916–7, Part II, **204**, p. 108.

Experiments * indicate that the value of the coefficient C is not constant, but that it diminishes as the ratio p_2/p_1 is increased approximately as indicated in the following table.

p_2/p_1	0·5	0·6	0·7	0·8	0·9	1·0
C	0·98	0·975	0·97	0·96	0·94	0·91

Measurement of Flow by Diaphragm in Pipe Line

The coefficients of discharge of standard sharp-edged orifices discharging freely are known with a fairly high degree of accuracy, and where such an orifice can be used for measuring the steady flow either of a liquid or of air, the results may be relied upon as being

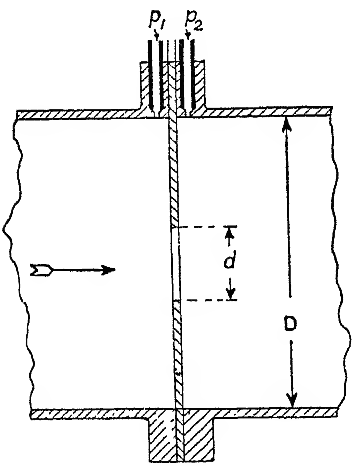

Fig. 13

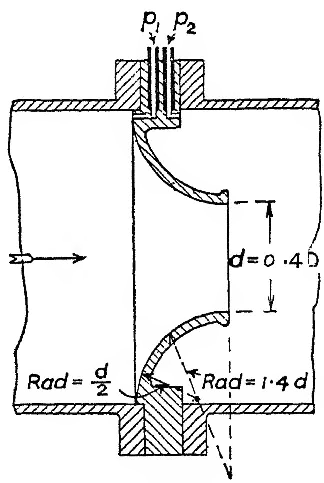

Fig. 14

accurate within 1 or 2 per cent, if suitable precautions are taken. Owing to the convenience of the method and the simplicity of the apparatus, much attention has recently been paid to the use of orifices through diaphragms in a pipe line for measuring the flow.

If D be the diameter of the pipe and d that of the orifice (fig. 13), Hodgson† states that the coefficient of discharge C for sharp-edged

* *Proc. Inst. Mech. E.*, Oct. 1919, p. 593.
† *Proc. Inst. C. E.*, 1916–17, Part II, p. 108.

orifice in a plate of thickness 0·02d, and for ratios of d/D less than 0·7 is 0·608 for water or air when p_2/p_1 is greater than 0·98, and is equal to $0{\cdot}914 - 0{\cdot}306\ p_2/p_1$ for air, steam, or gas when p_2/p_1 is less than 0·98, pressures being measured at the wall of the pipe immediately on each side of the diaphragm.

A rounded nozzle, if well designed, has a coefficient which varies from about 0·94 in small nozzles to 0·99 for large nozzles, either for water or air, if p_2/p_1 is greater than 0·6. Fig. 14* shows a form of nozzle in which the coefficient lies between 0·99 and 0·997.

The Pitot Tube

For measurements of the flow in pipes or in unconfined streams where the velocity is fairly high, the Pitot tube is capable of giving excellent results. This usually consists of a bent tube terminating in a small orifice pointing upstream, which is surrounded by a second tube whose direction is parallel to that of flow. A series of small holes in the wall of the outer tube admit water, at the mean pressure

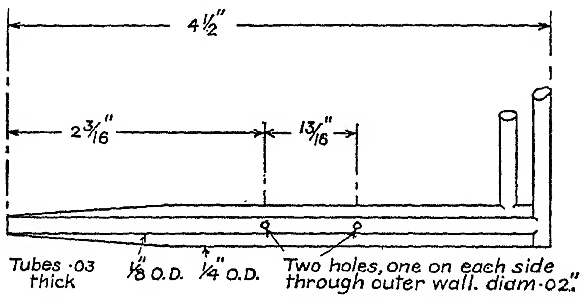

Fig. 15

in their vicinity, to its interior, which is connected to one leg of a manometer. The other leg is connected to the central tube carrying the impact orifice. If v is the velocity of flow immediately upstream from this orifice, the pressure inside the orifice, where the velocity is zero, is equal to the sum of the statical pressure at the point, plus $kv^2/2g$ ft. of water, where k is a constant whose value approximates closely to unity in a well-designed tube. It follows that the difference of level of the two legs of the manometer equals $kv^2/2g$.

Figs. 15, 16, and 17 show modern types of this instrument. Fig. 15 shows the type used for measuring the air speed of aeroplanes

* *Engineering*, 1st Dec., 1922, p. 690.

and for wind tunnel investigations. A tube of this type having the dimensions shown gave K = 1·00 within 1 per cent.*

The tube illustrated in fig. 16 gave a value of C = 0·926 when calibrated by towing through still water, and 0·895 when calibrated in a 2-in. pipe. The low value of C in still water is probably due to the fact that the pressure orifices are too near the shoulder of the pressure pipe. If this were lengthened, with the orifices farther back, the coefficient would probably be higher. The difference between the calibration in still water and in the small pipe is to be expected, since the velocity at the section of the pipe containing the pressure orifices is of necessity increased by the presence of the tube, and the pressure as recorded by the static pressure column will consequently be less than in the plane of the impact orifice. This effect will increase with the ratio of the diameter of tube to that of the pipe, and unless this ratio is small, the tube should be calibrated in a pipe of approximately the same dimensions as that in which it is to be used. It is preferable, for pipe work, to use a simple Pitot tube having only an impact orifice, and to obtain the static pressure from an orifice in the pipe walls in the plane of the impact orifice. Using the tube in this way, the coefficient C may usually be taken as 0·99, within 1 per cent.

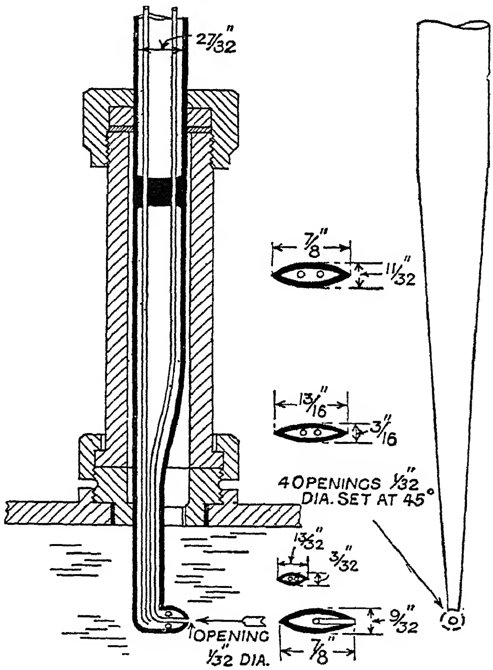

Fig. 16.—Details of Pitot Tube

In the type of Pitot tube shown in fig. 17, and known as the "Pitometer", the pressure at the downstream orifice is less than the

* "The Theory and Development of the Pitot Tube", Gibson, *The Engineer*, 17th July, 1914, p. 59.

statical pressure in the pipe and the coefficient is less than in the normal type. For the tube shown, calibrations in flowing water in pipes give a mean value of C = 0·916. Owing to eddy formation at the downstream orifice the coefficient of such a tube fluctuates within fairly wide limits.

The Pitot tube may be calibrated either in still water or in a current. In the latter case the mean velocity is computed from readings taken at a large number of points in a cross section, and

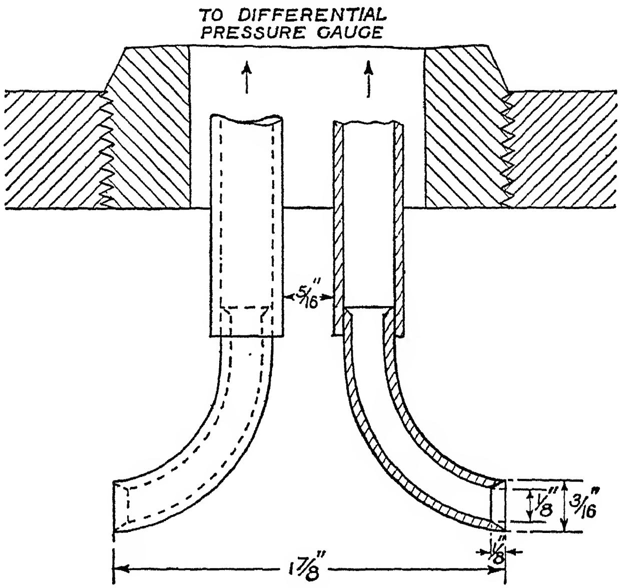

Fig. 17

the coefficient of the instrument is adjusted so as to make this mean velocity agree with that obtained from weir flow, current meter, or gravimeter measurements on the same stream.

Without exception, observers have found that a still-water rating gives a somewhat higher value of C. The explanation would appear to be twofold. In the first place, the velocity of flow in a moving current is never quite steady, but suffers a series of periodic fluctuations, and since the Pitot tube is an instrument which essentially measures the mean momentum, or the mean (v^2) of the flow and not its mean velocity, any such fluctuation superposed on a given mean velocity will give an increased head reading. In the second place, when metering a flowing current the average tube cannot be used at

points very near the boundary where the velocity is least, and for this reason also the mean recorded velocity tends to be too high.

It follows that although a still-water or still-air rating represents the true coefficient of the instrument, this requires to be reduced somewhat for use in a current, the effect increasing with the unsteadiness of the current. Where a high degree of accuracy is required, the rating should be carried out under conditions as nearly as possible resembling those under which measurements have afterwards to be made.

For measurements of the flow in pipes, the instrument should be used if possible at a section remote from any bend or source of disturbance. For approximate work the velocity of the central filament may be measured. This when multiplied by a coefficient which varies from 0·79 in small pipes to 0·86 in large pipes gives the mean velocity. Alternatively the velocity may be measured at the radius of mean velocity, which varies from $0{\cdot}7a$ in small pipes to $0{\cdot}75a$ in large pipes, where a is the radius of the pipe. These values, however, only apply to a straight stretch of the pipe, and if it is necessary to make measurements near a bend, and in any case for accurate results, the pipe should be traversed along two diameters at right angles, and the velocities measured at a series of radii. If δr is the width of an elementary annulus containing one series of such measurements whose mean value is v, the discharge is then given by

$$Q = \int_0^a 2\pi r v\,dr.$$

Several methods are available for determining the mean velocity of flow of a liquid in an open channel. This may be obtained:

(*a*) by the use of current meters giving the velocity at a series of points over a cross section of the channel;

(*b*) by the use of a standard weir;

(*c*) by the use of floats;

(*d*) by chemical methods. This method is best adapted to rapid and irregular streams, although it may be applied to the measurement of pipe flow. It consists in adding a solution of known strength of some chemical for which sensitive reagents are available, at a uniform and measured rate into a stream,* and by collecting and

* For a description of the method of introducing the solution uniformly, reference may be made to *Mechanical Engineering*, 44, April, 1922, p. 253, or to *Hydro-electric Engineering*, Gibson, Vol. I, p. 29.

analysing a sample taken from the stream at some lower point where admixture is complete. The solution should be added and the sample taken at a number of points distributed over the cross section. Various chemicals may be used. Unless the water is distinctly brackish, common salt is suitable. If brackish, sulphuric acid or caustic soda may be used. With a solution consisting of 16 lb. of salt per cubic foot of water, a dilution of 1 in 700,000 will give, on titration with silver nitrate, a precipitate weighing 1 mgm. per litre of the sample, and the gravimetric analysis of such a sample will enable an accuracy of 1 per cent to be attained.

If Q = discharge of stream to be measured in cubic feet per second,
q = quantity of solution introduced in cubic feet per second,
C_1 = concentration of salt in the natural stream water in pounds per cubic foot,
C_2 = concentration of salt in the sample taken downstream in pounds per cubic foot,
C = concentration of salt in the dosing solution in pounds per cubic foot,

then $$Q = \left(\frac{C - C_2}{C_2 - C_1}\right)q;$$

and if V_1, V_2, and V are the volumes of silver nitrate solution respectively necessary to titrate unit volume of normal stream water, of the downstream sample, and of the dosing solution,

$$Q = \left(\frac{V - V_2}{V_2 - V_1}\right)q.$$

(*e*) By injecting colour and by noting the time required for this to cover a measured distance.

A close approximation to the true discharge may be obtained either by the use of a weir, of current meters, or by chemical methods if suitable precautions are taken.* The colour method would not, in general, appear to be so reliable, and float measurements cannot be relied upon for any close degree of accuracy.

The Effect of Fluid Motion on Heat Transmission

Apart from the effect of radiation, the heat transmission between a solid surface and a fluid in motion over it will, for a given difference in temperature, be proportional to the rate at which the fluid

* See *Hydraulics*, Gibson, 1912 (Constable & Co.), p. 346.

particles are carried to and from the surface, and therefore to the diffusion of the fluid in the vicinity of the surface. Such diffusion depends on the natural internal diffusion of the fluid at rest, and on the eddies produced in turbulent motion which continually bring fresh particles of fluid up to the surface. In stream-line flow the second source of diffusion is absent; the heat transmission can only take place in virtue of the thermal conductivity of the fluid; and the rate of heat transmission is very small. Assuming that H, the heat transmitted per unit time per unit of surface, is proportional to θ, the difference of temperature between surface and fluid, the combined effect of the two causes may be written

$$\mathrm{H} = \mathrm{A}\theta + \mathrm{B}\rho v\theta, \qquad (8)$$

where ρ is the density of the fluid, A and B are constants depending on its nature, and v is its mean velocity.

As pointed out by Reynolds,* the resistance to the flow of a fluid through a tube may be expressed as

$$\mathrm{R} = \mathrm{A}'v + \mathrm{B}'\rho v^2, \qquad (9)$$

and various considerations lead to the supposition that A and B in (8) are proportional to A′ and B′ in (9). For assuming, as is now generally accepted, that even in turbulent flow there is a thin layer of fluid at the surface which is in stream-line motion, the heat transmission through this layer will be by conduction, and from the boundary of this layer to the main body of fluid by eddy convection. In stream-line flow the transfer of momentum which gives rise to the phenomena of viscosity is due to internal diffusion, while in turbulent motion the transference of momentum is due to eddy convection, so that it would appear that the mechanism giving rise to resistance to flow is essentially the same as that giving rise to heat transmission, both in stream-line and turbulent motion.

The following general explanation of the Reynolds law of heat transmission is due to Stanton.† Neglecting the effect of conductivity compared with that of viscosity, the ratio of the momentum lost by skin friction between two sections δx in. apart, to the total momentum of the fluid, will be the same as the ratio of the heat actually supplied by the surface, to that which would have been supplied if the whole of the fluid had been carried up to the surface.

* Reynolds, *Manchester Lit. and Phil. Soc.*, 1834.

† "Note on the Relation between Skin Friction and Surface Cooling", *Tech. Report, Advisory Committee for Aeronautics*, 1912–3.

Thus in pipe flow:

if δp is the difference in pressure at the two sections;
δT is the rise in temperature between the two sections;
W is the weight of fluid passing per second;
v_m is the mean velocity of flow;
T_m is the mean temperature of the fluid;
T_s is the temperature of the surface;
a is the radius of the pipe;

the above relationship becomes

$$\frac{\delta p(\pi a^2)}{\frac{W}{g}v_m} = \frac{W\delta T}{W(T_m - T_s)} \quad \ldots\ldots\ldots\ldots (10)$$

The heat gained per unit area of the pipe per second is

$$\frac{\sigma\frac{W}{g}\delta T}{2\pi a\delta x},$$

where σ is the specific heat. If R is the resistance per unit area,

$$R = \frac{\pi a^2}{2\pi a}\frac{\partial p}{\partial x},$$

so that if H is the heat transmitted per unit area per second,

$$H = \frac{R\sigma(T_s - T_m)}{v_m} \quad \ldots\ldots\ldots\ldots (11)$$

Since, as pointed out by Reynolds, the heat ultimately passes from the walls of the pipe to the fluid by conductivity, a correct expression for heat transmission should involve some function of the conductivity, and for this reason expression (11) can only be expected to give approximate results. In spite of this it enables some results of extreme practical value to be deduced. Thus if $R = kv^n$, and if σ be assumed constant,

$$H \propto v^{n-1}(T_s - T_m), \quad \ldots\ldots\ldots\ldots (12)$$

so that if the resistance be proportional to v^2, and if $T_s - T_m$ be maintained constant, the heat transmitted per unit area will be proportional to v, and since the mass flow is also proportional to v, the change in temperature of the fluid during its passage through

the pipe will be independent of the velocity. Otherwise, the heat transmitted will be directly proportional to the mass flow.

The general truth of this was demonstrated experimentally by Reynolds,* who showed that when air was forced through a hot tube, the temperature of the issuing air was sensibly independent of the speed of flow.

In the case of the flow of hot gases through the tubes of a boiler, or of the water through the tubes of a condenser, n is usually less than 2 and has a value of about 1·85. Moreover, in the former case any increase in the velocity of flow will be accompanied by an increase in the temperature of the metal surface, so that for both reasons the heat transmitted is not quite proportional to the mass flow, and the issuing gases are slightly hotter with a high velocity than with a low velocity of flow. The difference is, however, not great, and it appears that by increasing the velocity of flow of the fluid, the output of a steam boiler, or of a surface condenser, may be considerably increased without seriously affecting the efficiency. This is in general accordance with Nicholson's † investigations on boilers working under forced draught. These showed that by increasing the mass flow, the heat transmitted to the water was increased in almost the same proportion, while the temperature of the flue gases was only slightly increased.

Numerous other observers ‡ have verified the general truth of the relationship expressed in equation (12), p. 182, for the flow of liquids and gases through pipes. Its more general application to other cases of heat dissipation in a current still awaits experimental proof.

Experiments on the heat dissipation from hot wires of small diameter in an air current, show that this is proportional to $v^{0·5}$, which, if this relationship is correct, would indicate that the resistance should be proportional to $v^{1·5}$. This is not in agreement with the generally accepted result that the resistance is proportional to v^2. An examination of the experimental data shows, however, that the product of vd in the wires on which the heat measurements were made, was small, and an examination of the curve showing $R/\rho v^2 d^2$ plotted against vd/ν (fig. 4, Chap. V), shows that in this part of the

* *Memoirs, Manchester Lit. and Phil. Society*, 1872.

† "Boiler Economics and the Use of High Gas Speeds", *Trans. Inst. of Engineers and Shipbuilders in Scotland*, **54**; "The Laws of Heat Transmission in Steam Boilers", J. T. Nicholson, D.Sc., *Junior Institute of Engineers*, 1909.

‡ Stanton, *Phil. Trans. Roy. Soc.* A, **190**, 1897; Jordan, *Proc. Inst. Mech. Engineers*, 1909, p. 1317; Nusselt, *Zeitschrift des Vereines deutscher Ingenieure*, 23rd and 30th Oct., 1909.

range the curve is very steep, indicating that the resistance is proportional to a value of n much less than 2. Although the data are insufficient to indicate the exact value of n they do not, at all events, disprove the foregoing hypothesis.

The difficulty in forming any definite decision as to the general validity of the hypothesis arises from the fact that in most resistance experiments on cooling systems, it has been tacitly assumed that the resistance is proportional to v^2, and the published data usually give the average value of the coefficient of resistance based on this assumption. Thus the resistance of honeycomb radiators is known to be nearly proportional to v^2, while the heat transmission per degree difference of temperature is approximately proportional to $v^{0 \cdot 85}$.

Experiments by Stanton and by Pannell* show that while equation (11), p. 182, gives moderate results for air flow through pipes, the calculated results obtained with water as the fluid are very different from those deduced experimentally, as appears from Table I.

TABLE I

Pipe Dia., Cm.	Mean Vel., Cm. per Sec.	Mean Temperature. Surface, Deg. C.	Mean Temperature. Fluid, Deg. C.	Friction, Dynes per Sq. Cm.	Heat Transmitted, Calories per Sq. Cm. per Sec.	Value of $\frac{R\sigma(T_s - T_m)}{V_m}$	Fluid.
0·736	296	28·2	15·93	29·8	4·43	12·35	Water.
0·736	296	51·65	39·65	26·0	5·08	10·5	
1·39	123·2	47·2	20·96	50·6	5·36	10·8	
1·39	69·0	47·3	21·21	17·1	3·28	6·5	
4·88	940	36·2	22·7	3·18	0·0162	0·0109	Air.
4·88	1180	37·4	22·5	5·14	0·0205	0·0155	
4·88	1480	43·5	23·5	8·15	0·0300	0·0266	
4·88	2188	43·0	26·2	14·9	0·0369	0·0267	

It will be seen that in the case of air flow the heat transmission calculated from equation (11) is about 76 per cent of that observed, while for water the calculated value is twice as great as that observed.

* *Phil. Trans. Roy. Soc.* A, **190**; *Tech. Report, Advisory Committee for Aeronautics*, 1912–3.

Mr. G. I. Taylor * suggests that equation (11), p. 182, may be modified to take into account the effect of conductivity by assuming that there is a surface layer of thickness t, having laminar motion, through which heat is conveyed by conduction; that the velocity at the inner boundary of this layer is U, and the temperature T_1; and that between the centre and this layer heat transmission is due to eddy convection.

The temperature drop in the laminar layer, of conductivity k, is given by

$$T_s - T_1 = \frac{Ht}{k}.$$

If R is the resistance per unit area at the surface, this will be equal to the resistance to shear of the lamina.

$$\therefore\ R = \frac{\mu U}{t},$$

$$\text{so that}\quad T_s - T_1 = \frac{H\mu U}{kR} \quad \ldots\ldots\ldots\ldots\ldots\ldots (13)$$

By analogy with (11) the rate at which heat is transmitted from the layer by eddies is

$$H = \frac{R\sigma(T_1 - T_m)}{(v_m - U)} \quad \ldots\ldots\ldots\ldots\ldots (14)$$

Substituting for $\frac{H}{R}$ in (14) from (13) gives

$$\frac{T_1 - T_m}{T_s - T_1} = \frac{(v_m - U)}{U}\,\frac{k}{\sigma\mu}.$$

If $\frac{U}{v_m} = r$,

$$\frac{v_m - U}{U} = \frac{1-r}{r};\quad \frac{v_m - U}{v_m} = 1 - r;\quad \frac{T_s - T_m}{T_1 - T_m} = 1 + \frac{r}{1-r}\left(\frac{\mu\sigma}{k}\right).$$

$$\therefore\ H = R\sigma\frac{(T_1 - T_m)}{(v_m - U)}$$

$$= R\sigma\left(\frac{T_s - T_m}{v_m}\right)\left\{\frac{1}{1 + r\left(\frac{\mu\sigma}{k} - 1\right)}\right\} \quad \ldots\ldots\ldots (15)$$

This equation is identical with (11) if the quantity in brackets is

* Advisory Committee for Aeronautics, *Reports and Memoranda*, No. 272, 1916.

unity, i.e. if $\mu\sigma = k$. For air this is very nearly the case, since $k = 1{\cdot}6\mu C_v$ where C_v is the specific heat at constant volume, and $C_v = \dfrac{\sigma}{1{\cdot}4}$, so that $\dfrac{\mu\sigma}{k} = 0{\cdot}88$. In the case of water, however, at $20°$ C., $\mu = 0{\cdot}01$, $k = 0{\cdot}0014$, $\sigma = 1{\cdot}0$, and $\dfrac{\mu\sigma}{k} = 7{\cdot}1$.

Stanton * has shown that the value of r necessary to bring the results as found from equation (15), p. 185, into line with the experimental results for water quoted in Table I, p. 184, is 0·29, and that similar experiments by Soenneker † require a mean value of 0·34. Taylor, from an examination of data by Lorentz,‡ concludes that the ratio is approximately 0·38. Some idea of its value in the case of air may be deduced from direct measurements by Stanton, Marshall, and Bryant,§ of the velocities in the immediate vicinity of the pipe wall. These measurements would appear to indicate that true laminar flow is instituted at a point where $\dfrac{U}{v_m}$ is approximately 0·14. They show, however, that it is erroneous to assume that at this point the change to true turbulent flow is abrupt, but that the change is gradual over an appreciable radial depth of the fluid. It follows that equation (15) has not a strictly rational basis, but that by assuming $r = 0{\cdot}30$ it gives results which are, for practical purposes, not seriously in error.

The ratio

$$\frac{\text{drop in temp. in surface layer}}{\text{drop in temp. in rest of tube}} = \frac{T_s - T_1}{T_1 - T_m} = \left(\frac{r}{1 - r}\right)\frac{\mu\sigma}{k}.$$

Thus, if the effective value of $r = 0{\cdot}30$, the ratio is 3·0 for water at $20°$ C., and 0·38 for air.

Reynolds ‖ has shown by an application of the principle of dynamical similarity that in the case of pipe flow

$$\frac{\partial p}{\partial x} = \frac{\nu^{2-n}}{d^{3-n}}\, v_m{}^n\, \frac{B^n}{A},$$

* *Dictionary of Applied Physics*, Vol. I, p. 401.

† *König Tech. Hochschule*, Munich, 1910.

‡ *Abhandlung über theoretische Physic*, Band, I, p. 343.

§ In air flow $R = 0{\cdot}002\rho V_m{}^2$ approximately, and Stanton's experiments (*Proc. Roy. Soc.* A, 1920) indicate that t is approximately 0·005 cm. when $\mu = 0{\cdot}00018$ and $V_m = 1850$ cm. per second. This makes $U = 0{\cdot}14V_m$.

‖ Chapter V.

and if this value of $\frac{\partial p}{\partial x}$ be used in (10), p. 182, on writing $W = \pi r^2 \rho v$, the expression becomes

$$\frac{dT}{dx} = \frac{B^n}{A} \frac{g}{\rho} \frac{\nu^{2-n}}{d^{3-n}} v_m^{n-2}(T_s - T), \ldots\ldots\ldots(16)$$

where now T is the temperature of the fluid, and $\frac{dT}{dx}$ is the temperature gradient along the pipe.

If T_s is sensibly constant along the pipe, integration of (16) gives

$$\log_e \frac{T_s - T_1}{T_s - T_2} = \frac{B^n}{A} \frac{g}{\rho} \frac{\nu^{2-n}}{d^{3-n}} v_m^{n-2} l, \ldots\ldots\ldots(17)$$

where l is the length of the pipe, and T_1 and T_2 are the temperatures of the fluid at inlet and outlet.

Stanton,* from experiments on heat transmission from water to a cold tube and vice versa, deduced the expression, for small values of $T_1 - T_2$,

$$\log_e \frac{T_s - T_1}{T_s - T_2} = k \frac{\nu^{2-n}}{d^{3-n}} v_m^{n-2} l \ \{(1 + \alpha T_s)(1 + \beta T_m)\}, \ldots(18)$$

where, in C.G.S. units, $\alpha = 0{\cdot}004$ and $\beta = 0{\cdot}01$. It will be noted that this expression is identical in form with (17), except for the last two factors, which were introduced to take into account the effect of the variation in conductivity, with temperature, of the surface film of water. In those experiments in which the heat flow was from metal to water, k had a mean value of 0·0104. With flow in the other direction k was, however, distinctly less, having a mean value of approximately 0·0075.

Application of the Principle of Dimensional Homogeneity to Problems involving Heat Transmission

The principle of dimensional homogeneity, Chap. V,† may readily be extended to problems involving heat transmission. In this case, in addition to the three fundamental mechanical units, a thermal unit is needed to define all the quantities involved. Taking temperature T as this unit, the new quantities, heat flow H, conductivity k,

* *Trans. Roy. Soc.* A, 1897.

† See also a note by Lord Rayleigh, *Nature*, 95, 1915, p. 66.

and specific heat σ which are now involved, may be expressed dimensionally as

$$H = \begin{Bmatrix}\text{heat flow per}\\ \text{unit time}\end{Bmatrix} = \begin{Bmatrix}\text{energy per}\\ \text{unit time}\end{Bmatrix} = ML^2t^{-3}.$$

$$k = \left\{\frac{H \times \text{length}}{\text{sectional area} \times T}\right\} = \frac{H}{LT} = MLt^{-3}T^{-1}.$$

$$\sigma = \left\{\frac{\text{heat per unit mass}}{\text{rise in temperature}}\right\} = \frac{Ht}{MT} = L^2t^{-2}T^{-1}.$$

If attention be confined to the large class of problems of practical importance, involving the transmission of heat between a fluid and a surface moving with relative velocity, where temperature differences are so small—not exceeding a few hundred degrees—that radiation is only of secondary importance, the only quantities involved are

$$H,\ T,\ k,\ \sigma,\ v,\ l,\ \rho,\ \mu.$$

We select l, v, ρ and σ as the four independent quantities, and combine them with the other four H, T, k and μ, in turn, so as to obtain $8 - 4$ $(=4)$ dimensionless quantities K, as explained at p. 198; i.e. we write $H = l^x v^y \rho^z \sigma^n$, and similarly with T, k and μ. We thus find

$$K_1 = \frac{H}{l^2v^3\rho};\ K_2 = \frac{\sigma T}{v^2};\ K_3 = \frac{k}{lv\rho\sigma};\ K_4 = \frac{\mu}{lv\rho};$$

$$\text{whence} \qquad \psi\left(\frac{H}{l^2v^3\rho},\ \frac{\sigma T}{v^2},\ \frac{k}{lv\rho\sigma},\ \frac{\mu}{lv\rho}\right) = 0;$$

$$\text{or} \qquad H = \rho l^2 v^3 \phi\left(\frac{\sigma T}{v^2},\ \frac{k}{lv\rho\sigma},\ \frac{\mu}{lv\rho}\right),$$

which, by combining the last two terms, becomes

$$H = \rho l^2 v^3 \phi\left(\frac{\sigma T}{v^2},\ \frac{k}{\sigma\mu},\ \frac{\mu}{lv\rho}\right) \qquad \text{(19)}$$

At such speeds as are usual in the case of air flow over air-cooled engine cylinders, of the flow of gases through boiler flues, or of heating or cooling liquids through pipes, experiment shows that the heat flow is sensibly proportional to v^n, where n is between 0·5 and 1·0, its value depending on the type of flow and the form of surface. Experiment, moreover, indicates that if radiation be neglected the heat flow is directly proportional to the difference of temperature

between the fluid and the surface, in which case the function ϕ in (19) must be of the form $\frac{\sigma T}{v^2}F\left(\frac{k}{\sigma\mu}, \frac{\mu}{lv\rho}\right)$, and (19) becomes

$$H = \rho l^2 v\sigma T F\left(\frac{k}{\sigma\mu}, \frac{\mu}{lv\rho}\right). \quad \ldots\ldots\ldots\ldots(20)$$

If, in addition, $H \propto v^n$, F must be of the form $\left(\frac{\mu}{lv\rho}\right)^{1-n} f\left(\frac{k}{\sigma\mu}\right)$, and (20) becomes

$$H = \rho^n l^{1+n} v^n \mu^{1-n} \sigma T f\left(\frac{k}{\sigma\mu}\right). \quad \ldots\ldots\ldots\ldots(21)$$

If the fluid to which k, σ and μ belong is a gas, $f\left(\frac{k}{\sigma\mu}\right)$ is a constant, by the kinetic theory of gases. In this case, we may take for $f\left(\frac{k}{\sigma\mu}\right)$ either $A\frac{k}{\sigma\mu}$ or $B\left(\frac{k}{\sigma\mu}\right)^{1-n}$, these being constants; thus obtaining the alternative forms for H,

$$H \propto l^{1+n}\left(\frac{v\rho}{\mu}\right)^n kT,$$

$$\text{or} \qquad H \propto l^{1+n}(v\rho\sigma)^n k^{1-n} T. \quad \ldots\ldots\ldots\ldots(22)$$

If F is the total resistance to the steady motion of the fluid, then since the heat loss per degree difference of temperature, per unit specific heat, is of dimensions

$$\frac{H}{\sigma T} = \frac{ML^2t^{-3}}{L^2t^{-2}T^{-1}\,.\,T} = Mt^{-1},$$

while F is of dimensions MLt^{-2}, the ratio $F\Big/\frac{H}{\sigma T}$ is of dimensions $L/t = v$, so that the index n in $H \propto v^n$ should be less by unity than the corresponding index in $F \propto v^{n'}$. From this it appears that with flow so turbulent as to give the n^2 law of resistance, n in equation (22) becomes 1, and

$$H \propto l^2 v\rho\sigma T, \quad \ldots\ldots\ldots\ldots\ldots\ldots(23)$$

while with stream-line flow ($n' = 1$ and $n = 0$),

$$H \propto lkT. \quad \ldots\ldots\ldots\ldots\ldots\ldots(24)$$

For example, in the case of flow through similar pipes, where the l term may be taken to represent the diameter, equation (23) indicates that in such circumstances H is independent of the conductivity of the fluid. Also since the weight of fluid W passing

a given section per second is proportional to the product $d^2 v\rho$,

$$H \propto W\sigma T.$$

It follows that in similar pipes the heat transmission per unit degree difference of temperature between wall and fluid is proportional to the weight of fluid passing, or in other words, that with a given inlet temperature the outlet temperature is independent of the weight of fluid.

If, however, n is somewhat less than 1, as is usually the case in practice, equation (22) shows that

$$H \propto W \frac{\sigma^n k^{1-n}}{(dv\rho)^{1-n}} T$$

$$\propto W^n \sigma^n (kd)^{1-n} T,$$

so that with a given pipe and fluid, the heat transmission does not increase quite so fast as the mass flow, and the outlet temperature will increase somewhat as the flow is increased.

CHAPTER V

Hydrodynamical Resistance

A body in steady motion through any real fluid, or at rest in a moving current, experiences a resistance whose magnitude depends upon the relative velocity, the physical properties of the fluid, the size and form of the body, and, at velocities above the critical, also upon its surface roughness.

At velocities below the critical, where the flow is " stream line ", the resistance is due essentially to the viscous shear of adjacent layers of the fluid. It is directly proportional to the velocity, to the viscosity, and, in bodies of similar form, to the length of corresponding dimensions. Thus the resistances to the motion of small spheres at such velocities are proportional to their diameters.*

With stream-line motion there is no slip at the boundary of solid and fluid, and the physical characteristics of the surface do not affect the resistance.

At velocities above the critical, where the motion as a whole is definitely turbulent, there would still appear to be a layer of fluid in contact with the surface in which the motion is non-turbulent.† The thickness of this layer is, however, very small, and any increase in the roughness of the surface, by increasing the eddy formation, increases the resistance. At such velocities the resistance is due in part to the viscous shear in this surface layer, but mainly to eddy formation in the main body of fluid. This latter component of the resistance depends solely on the rate at which kinetic energy is being given to the eddy system, and is proportional to the density of the fluid and to the square of the velocity.

Although the viscosity of a fluid provides the mechanism by which eddy formation becomes possible, and by which the energy of the eddies, when formed, is dissipated in the form of heat, it has only a very small effect on the magnitude of the resistance in turbulent motion, and, as will be shown later, it can have no direct effect in

* See H. S. Allen, *Phil. Mag.*, September and November, 1900.

† Stanton, *Proc. Roy. Soc.* A, 97, 1920.

a system in which the resistance is wholly due to eddy formation, and in which the resistance is, in consequence, proportional to v^2.

Experiments carried out over a limited range of velocities have usually shown that with turbulent flow the resistance of any given body is proportional to v^n, where n is slightly less than 2, although experiments on flow in rough pipes, on the resistance of cylinders, of inclined plane surfaces, and of air-ship bodies, show that in such cases the variation from the index 2 may be within the limits of experimental error. With smooth pipes, however, n may be as low as 1·75, and with ship-shaped bodies of fair form in water is usually about 1·85.

More recent experiments * indicate that no one constant value of n holds over a very wide range of velocities, but that n increases with the velocity, and that a formula of the type

$$R = Av + Bv^2 + Cv^3,$$

where A and B and C are constants depending upon the form and roughness of the body and on the physical properties of the medium, more nearly represents the actual results. Over a moderate range of velocities a single value of n can usually be obtained which gives the resistance, within the errors of observation, and in view of the convenience of such an exponential formula it is commonly adopted in practice.

At velocities above the critical, the direct influence of viscosity increases with the departure of the index n from 2. When $n = 2$ the resistance is proportional to the density of the fluid, and, in similar bodies, to the square of corresponding linear dimensions.

Between the low velocities at which the motion is stream-line, and the high velocities at which it is definitely turbulent, there is a range over which it is extremely unstable, and in which the resistance may be affected considerably by small modifications in the form, presentation, or surface condition of the body. Thus the resistance of a sphere, at a certain velocity whose magnitude depends on the diameter, is actually increased instead of being diminished by reducing its roughness.

In problems occurring in practice, however, velocities are in general well above the critical point. One noteworthy exception is to be found in the flow of oil through pipe lines in which, owing to the high viscosity of the fluid, the motion is usually non-turbulent.

In hydrodynamical problems it is usual to assume that the

* N.P.L., *Collected Researches*, **11**, 1914, p. 307.

resistance depends solely on the relative velocity of fluid and body, and that it is immaterial whether the body is at rest in a current of fluid, or is moving through fluid at rest. Although there is not much direct experimental evidence on this point, it is probable that while with stream-line motion the resistance is identical in both cases, in turbulent motion it is not necessarily so, and that it may be sensibly greater when the fluid is in motion than when the body is in motion.

This is to be expected when it is realized that in a fluid in motion with a mean velocity v, many of its particles have a higher velocity, so that the kinetic energy is greater than that given by the product of the mass and the square of the mean velocity. Any difference arising from this effect will in general only be small, but comparatively large differences may be expected over the range of velocities in which the motion is unstable, owing to the fact that, with a stationary body, the interaction occurs in a medium which has an initial tendency to instability owing to its motion.

Thus the system of eddy formation in the rear of any solid body advancing into still water may reasonably be expected to differ from that behind the same body in a current of the same mean velocity, owing to the instability in the latter case of the medium in which, and from which in part, it is being maintained.

Except in the case of stream-line flow, the laws of hydrodynamical resistance can only be deduced experimentally. Much information can, however, be obtained regarding these laws from an application of the two allied principles of dynamical similarity and dimensional homogeneity.

Dynamical Similarity

Two systems, involving the motion of fluid relative to geometrically similar bodies, are said to be dynamically similar when the paths traced out by corresponding particles of the fluid are also geometrically similar and in the same scale ratio as is involved in the two bodies.

The densities of the fluids may be different, as also the velocities with which corresponding particles describe their paths. If the densities in the two systems are in a constant ratio, and the velocities of corresponding particles are also in a constant ratio, then the ratio of corresponding forces can be determined. In fact, the scale ratio of velocities and that of lengths being both given, the scale ratio of times is determined, and therefore also the scale ratio of accelerations. By means of the fundamental relation, force = mass × acceleration, the ratio of corresponding forces can then be found.

In two systems, denoted by (1) and (2), if w be the weight of unit volume, ρ the density, v the velocity, l any definite linear dimension, and r the radius of curvature of the path, these forces are in the ratio

$$\frac{F_1}{F_2} = \frac{w_1 v_1^2}{r_1} \frac{r_2}{w_2 v_2^2} = \frac{\rho_1 l_1^3 r_2}{\rho_2 l_2^3 r_1} \left(\frac{v_1}{v_2}\right)^2$$
$$= \frac{\rho_1}{\rho_2} \frac{l_1^2}{l_2^2} \left(\frac{v_1}{v_2}\right)^2.$$

It follows that for dynamical similarity corresponding velocities must be such as to make the corresponding forces due to each physical factor proportional to $\rho l^2 v^2$. The velocities so related are termed " Corresponding Speeds ".

Where the only physical factor involved is the weight of the fluid, since the force due to this is proportional to ρl^3, the required condition will evidently be satisfied if $\dfrac{v_1}{v_2} = \sqrt{\dfrac{l_1}{l_2}}$.

On the other hand, if viscous forces are all important, since the force due to viscosity equals $\mu \dfrac{dv}{dl}$ per unit area, where μ is the coefficient of viscosity,

$$\frac{F_1}{F_2} = \frac{\mu_1}{\mu_2} \frac{l_1^2}{l_2^2} \frac{v_1}{v_2} \frac{l_2}{l_1}$$
$$= \frac{\mu_1}{\mu_2} \frac{l_1}{l_2} \frac{v_1}{v_2},$$

and for this to equal $\dfrac{\rho_1}{\rho_2} \dfrac{l_1^2}{l_2^2} \left(\dfrac{v_1}{v_2}\right)^2$ it is necessary that

$$\frac{\rho_1 l_1 v_1}{\mu_1} = \frac{\rho_2 l_2 v_2}{\mu_2},$$

$$\text{or that} \quad \frac{v_1}{v_2} = \frac{\mu_1 \rho_2 l_2}{\mu_2 \rho_1 l_1} = \frac{\nu_1 l_2}{\nu_2 l_1},$$

where ν is the " kinematic viscosity " μ/ρ.

Generally speaking, wherever the influence of gravity is involved in the interaction between a solid and a fluid, as is the case where surface waves or surface disturbances are produced, and where the direct influence of viscosity is negligible, corresponding speeds are proportional to the square roots of corresponding linear dimensions; while where gravitational forces are not involved and where the forces are due to viscous resistances, corresponding speeds are

inversely proportional to corresponding linear dimensions, and directly proportional to the kinematic viscosities.

The flow of water from a sharp-edged orifice under the action of gravity is an example of the first type of interaction, while the resistance of an air-ship, or of a submarine submerged to such a depth that no surface waves are produced, is representative of the second type.

The resistance of a surface vessel is one of a series of typical cases, of importance in practice, in which both gravity and viscosity are involved, and in which therefore no two corresponding speeds will satisfy all requirements. In other words, the speeds which give geometrically similar wave formations around two similar ships will not give similar stream-lines in those parts of the systems subject to viscous flow. If, however, the influence of one of these factors is much greater than that of the other, approximately similar results, which may be of great value in practice, can be obtained by using corresponding speeds chosen with reference to the important factor. Thus in tank experiments on models of floating vessels the corresponding speeds are chosen with reference to the wave and eddy effects, and are proportional to the square root of corresponding linear dimensions. This involves a scale error for which a correction is made as explained on p. 213.

Dimensional Homogeneity

In view of the value of the results which may be obtained by the use of the principle of dimensional homogeneity in problems involving fluid resistance, the method of its general application will now be outlined.

The principle of dimensional homogeneity states that all the terms of a correct physical equation must have the same dimensions. That is, if the numerical value of any one term in the equation depends on the size of one of the fundamental units, every other term must depend upon it in the same way, so that if the size of the unit is changed, every term will be changed in the same ratio, and the equation will still remain valid.

The quantities which occur in hydrodynamics may all be defined in terms of three fundamental units. The most convenient units are usually those of mass (M), length (L) and time (t).

Example 1.—Suppose some physical relationship to involve only four quantities, say a force R, a length l, a velocity v, and a density ρ.

Let it be assumed provisionally that the relationship is of the form

$$R \propto l^x v^y \rho^z.$$

Expressed dimensionally, this gives

$$MLt^{-2} = L^x \,.\, L^y t^{-y} \,.\, M^z L^{-3z},$$

and, on equating the indices of like quantities,

$$1 = z,\ 1 = x + y - 3z,\ -2 = -y;$$

whence $$z = 1,\ y = 2,\ x = 2.$$

It follows that $R \propto l^2 v^2 \rho$, provided the initial assumption as to the form of R is correct.

It is possible, however, to obtain the result without making this assumption. What we have really proved, in fact, is that the quantity

$$\frac{R}{l^2 v^2 \rho}$$

is dimensionless. Also, since it is given that there is a relationship between R, l, v, ρ, the quantity $R/l^2v^2\rho$ is certainly some function of l, v, ρ, say $f(l, v, \rho)$. Now, since the units of l, v, ρ are independent we can, by changing the unit of l, say, change the numerical value of l without changing the numerical values of v or ρ. This change does not change the value of $f(l, v, \rho)$, since it does not change the value of the dimensionless number $R/l^2v^2\rho$, to which $f(l, v, \rho)$ is equal. Hence the function f does not involve l, and similarly it does not involve v or ρ; it is therefore a mere numerical constant, so that $R \propto l^2v^2\rho$.

Example 2.—If more than four quantities of different kinds are involved, for example R, l, v, ρ, μ, where μ is a viscosity (dimensions $ML^{-1}t^{-1}$, p. 28), the assumption $R \propto l^x v^y \rho^z \mu^p$ would not allow us to determine the values of x, y, z, p by considerations of dimensions, since there would be only three equations in four unknowns. It is possible, however, to obtain a new dimensionless number, not involving R, but of the form

$$\frac{\mu}{l^a v^b \rho^c}.$$

Equating the dimensions of μ to those of $l^a v^b \rho^c$, we find

$$1 = c,\ -1 = a + b - 3c,\ -1 = -b;$$

and $$c = 1,\ b = 1,\ a = 1.$$

Thus $\mu/lv\rho$ is dimensionless.

For brevity, we shall denote the two dimensionless numbers now found by K_1, K_2; i.e.

$$\frac{R}{l^2v^2\rho} = K_1, \qquad \frac{\mu}{lv\rho} = K_2. \quad \ldots\ldots\ldots\ldots(1)$$

It will now be proved that if there is a relationship between R, l, v, ρ, μ it can be expressed in the form

$$K_1 = f(K_2), \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

where the form of the function f remains undetermined. In fact, since R by assumption is some function of l, v, ρ, μ, and since we can substitute $lv\rho \cdot K_2$ for μ, it follows that $R/l^2v^2\rho$ is some function of l, v, ρ, K_2, or say

$$K_1 = \phi(l, v, \rho, K_2).$$

Then exactly the same argument as that given under Ex. 1 proves that the function ϕ does not involve l or v or ρ, but only K_2, and this is what was to be proved. The relation $K_1 = f(K_2)$ may of course be written in other forms, as for example $K_2 = F(K_1)$ or $\psi(K_1, K_2) = 0$.

The general theorem of dimensionless numbers.

The general theorem, of which the two preceding results are particular cases, may be stated as follows:*

(1) Let it be assumed that n quantities $Q_1, Q_2, \ldots, Q_n$ which are involved in some physical phenomenon, are connected by a relation,

$$F(Q_1, Q_2, \ldots, Q_n) = 0, \ldots\ldots\ldots\ldots\ldots(3)$$

containing these quantities and nothing else but pure numbers.

(2) Let k be the number of fundamental units (L, M, t, ...) required to specify the units of the Q's.

(3) Let $Q_1, Q_2, \ldots, Q_k$ be any k of the Q's that are of independent kinds, no one being derivable from the others, so that these k might, if we so desired, be taken as the fundamental units.

(4) Let Q_x be any one of the remaining $n - k$ quantities Q, and let $\frac{Q_x}{Q_1{}^a Q_2{}^b \ldots Q_k{}^p}$, which we denote by K_x, be the dimensionless quantity formed from Q_x and powers of $Q_1, \ldots, Q_k$.

*E. Buckingham, *Physical Review*, IV, 1914, p. 345; *Phil. Mag.*, Nov., 1921, p. 696.

(5) Then the theorem is that the equation

$$F(Q_1, Q_2, \ldots, Q_n) = 0$$

is reducible to the form

$$\phi(K_1, K_2, \ldots, K_{n-k}) = 0, \quad \ldots\ldots\ldots\ldots (4)$$

or, alternatively,

$$K_1 = f(K_2, K_3, \ldots, K_{n-k}). \quad \ldots\ldots\ldots\ldots (5)$$

The proof follows exactly the same lines as in the two particular examples already given.

The actual forms of the functions ϕ and f can only be deduced from experiment.

In the most general case of a dynamical relationship between any number of quantities, say n, there will be $n - 3$ quantities (K) of zero dimensions, composed of powers of the n quantities, and deduced in the manner explained above. The relationship between the n quantities originally considered reduces to a relationship between the $n - 3$ quantities $K_1, K_2, \ldots, K_{n-3}$. Hence if all but one of the K's are known, the last one is determined.

Some of the K's may often be written down from inspection. Suppose, for example, that a certain phenomenon involves a time t and an acceleration g, in addition to the R, l, v, ρ, μ of Ex. 2 above. Then we see at once that the new K's are tv/l and gl/v^2. The relation between the seven quantities is therefore of the form

$$\psi\left(\frac{R}{l^2v^2\rho}, \frac{\mu}{lv\rho}, \frac{tv}{l}, \frac{gl}{v^2}\right) = 0, \quad \ldots\ldots\ldots\ldots (6)$$

or, say,

$$R = l^2v^2\rho\,\phi\left(\frac{\mu}{lv\rho}, \frac{tv}{l}, \frac{gl}{v^2}\right). \quad \ldots\ldots\ldots\ldots (7)$$

It is useful to remark that any product of powers of the K's is dimensionless. Hence if we multiply the second and third of the arguments of ϕ in (7), we get a new dimensionless number tg/v, which may perfectly well replace one of the two, tv/l and gl/v^2, in (6) and (7).

If two or more quantities of the same kind are involved, as, for example, in the case of the resistance of an airship body, where both the length and diameter of the body affect the result, these may be specified by the value of any one, and by the ratios of the others to this one. Thus, in the problem just considered, if besides

R, v, ρ, μ, g, t and the length l of a body, there are also involved the breadth b and the depth d of the body, the solution is

$$R = l^2 v^2 \rho F\left(\frac{\mu}{lv\rho}, \frac{tg}{v}, \frac{lg}{v^2}, \frac{b}{l}, \frac{d}{l}\right). \quad \ldots\ldots\ldots (8)$$

It is clear from the above examples that the actual arithmetic involved in working out an application of the principle of dimensions is of the simplest possible kind. The real difficulty is in making sure that all the essential quantities concerned in the phenomenon are being taken into account. If this preliminary condition is not fulfilled, the result obtained will be quite erroneous.

Resistance to the Uniform Flow of a Fluid through a Pipe

An examination of the factors involved in the non-accelerated motion of a fluid through a pipe would indicate that the pressure drop p/l per unit length of the pipe may depend in some way on the diameter d, the velocity of flow, and the density and viscosity of the fluid. In a liquid where the effect of elasticity is negligible, it is difficult to imagine any other factor likely to affect the pressure drop, except the roughness of the pipe walls; and if this roughness is proportional to the diameter, the general unreduced relationship is of the form

$$F(d, v, \rho, \mu, p/l) = 0. \quad \ldots\ldots\ldots\ldots\ldots (9)$$

Here the number of dimensionless quantities K is $5 - 3 = 2$. Exactly as in a former example (p. 196) we find

$$K_1 = \frac{pd}{l\rho v^2}, \qquad K_2 = \frac{\mu}{dv\rho}.$$

There is some advantage in working with ν, the kinematic viscosity, which is equal to μ/ρ, rather than with μ itself. Hence $K_2 = \nu/dv$.

The reduced relationship may therefore be written

$$\frac{p}{l} = \frac{\rho v^2}{d}\,\phi\left(\frac{\nu}{dv}\right), \quad \ldots\ldots\ldots\ldots\ldots\ldots (10)$$

where the form of the function ϕ remains to be found from experiment. Note that the value of the function ϕ for all values of its argument can be found by varying *one only* of ν, d, v.

With stream-line flow, experiment shows that $\frac{p}{l}$ is proportional to v. It follows that $\phi\left(\frac{\nu}{dv}\right)$ must equal $\frac{k\nu}{vd}$, and that

$$\frac{p}{l} = \frac{k\mu v}{d^2},$$

where k is a numerical coefficient. This is Poiseuille's expression for the resistance to viscous flow, the coefficient k having the value 32.*

If the flow is turbulent the pressure gradient is approximately proportional to v^n, where n is between 1·75 and 2·0. In this case $\phi\left(\frac{\nu}{dv}\right)$ must be such as to make $\phi\left(\frac{1}{v}\right) = v^{n-2}$, so that

$$\frac{p}{l} = \frac{\rho v^2}{d}\left(\frac{\nu}{dv}\right)^{2-n}$$

$$= k'\frac{\rho v^n \nu^{2-n}}{d^{3-n}}, \quad\ldots\ldots\ldots\ldots\ldots\ldots(11)$$

$$\text{or} \qquad h = \frac{k' v^n \nu^{2-n}}{d^{3-n}} l, \quad\ldots\ldots\ldots\ldots\ldots\ldots(12)$$

where h is the difference of head at two points distant l apart, expressed as a length of a column of the fluid. This is the Reynolds† formula for pipe flow.

If the foregoing assumptions are correct, on plotting observed values of $\frac{pd}{lv^2\rho}$ against simultaneous values of $\frac{\nu}{dv}$, in any series of experiments in which different liquids or pipes of different diameters are used, the points should lie on a single curve.

That this is the case for fluids so widely different as air, water, and oil, has been shown by various observers, notably by Stanton and Pannell‡ (fig. 1). The experimental points for both air and water lie evenly about the two curves shown, the upper referring to a rough pipe, and the lower to a smooth pipe. The agreement in the case of air can, however, only be expected to be close where the drop in pressure is so small that the effect of the change of density

* Gibson, *Hydraulics and its Applications* (Constable & Co., 1912), p. 69.

† *Scientific Papers*, Osborne Reynolds, Vol. II, p. 97.

‡ *Phil. Trans. Roy. Soc.* A, **214**, 1914, p. 199.

is negligible. For large changes of pressure it may be shown* that formula (12) becomes

$$\frac{\delta p}{l} = \frac{kv'^{n}_{m}\mu^{2-n}}{d^{3-n}}\left[\frac{p_m}{\mathrm{CT}}\right]^{n-1} \quad \ldots\ldots\ldots\ldots(13)$$

Here T is the absolute temperature of the gas, C is the constant obtained from the relationship $pV = CT$,† and v_m and p_m are the mean velocity and pressure in the pipe. The results of experiments on the flow of air through pipes by several experimenters, with dia-

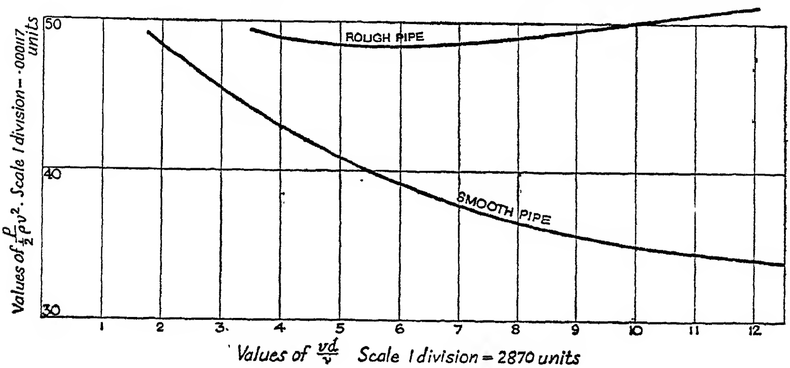

Fig. 1

meters ranging from 0·125 in. to 0·91 ft., and at velocities from 10 to 40 ft.-sec., confirm the accuracy of this formula.

Below the critical velocity, $n = 1$, and the formula becomes

$$\frac{\delta p}{l} = \frac{k\mu v_m}{d^2},$$

showing that the pressure drop is now independent of the absolute pressure in the pipe, a result confirmed by experiment.

Equations (10), p. 191, (11), p. 200, show that, with an incompressible fluid, if the resistance to flow varies as v^2, $\phi\left(\frac{\nu}{dv}\right)$ degenerates into a numerical constant. Viscosity ceases to have any direct influence on the resistance,‡ and true similarity of flow should be obtained in all pipes at all velocities. If n is less than 2, the

* Gibson, *Phil. Mag.*, March, 1909, p. 389.

† In the case of air, if the mass is unity, and if p be measured in pounds weight per square foot, the value of C is $53{\cdot}18 \times 32{\cdot}2 = 1710$; while if p be measured in pounds per square inch C becomes 11·9.

‡ See footnote on p. 209.

equations show that for similarity of flow in two pipes of different diameters, or conveying different fluids, it is necessary that $\frac{\nu}{vd}$ should be the same in both cases.

This has been shown to be true over a moderate range of diameters by Stanton.* who measured the distribution of velocity with turbulent flow across the diameters of two rough pipes of different diameters and repeated the measurements for two smooth pipes. In the rough pipes ($n = 2$) the velocity distribution (CC′, fig. 2) was the same in both pipes and at all

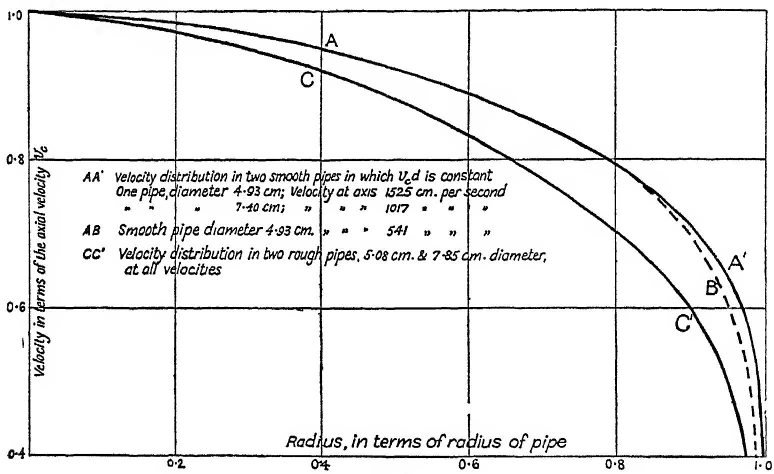

Fig. 2

velocities. In the smooth pipes ($n < 2$) identical curves were obtained only when $\frac{\nu}{vd}$ was the same in each pipe. Under other conditions the curves were sensibly identical from the centre up to a radius of about 0·8 times the radius of the pipe, but differed appreciably at larger radii (AA′ and AB, fig. 2).

In any type of pipe the "critical velocity" at which the type of motion changes from stream-line to turbulent is obtained with a constant value of $\frac{\nu}{vd}$, and this is generally true for fluid motion under other circumstances.

As already indicated, experiment shows that the resistance to

* *Proc. Roy. Soc.* A, 85, 1911.

flow in a smooth pipe where n is less than 2 is not strictly proportional to any one power of the velocity, and Dr. C. H. Lees* has shown that Stanton and Pannell's results for smooth pipes are very closely represented by the empirical relationship

$$\frac{p}{l} = \frac{\rho v^2}{d}\left\{a\left(\frac{\nu}{vd}\right)^{\alpha} + b\right\},$$

where $\alpha = 0{\cdot}35$ and a and b are constants, so that

$$\phi\left(\frac{\nu}{vd}\right) = a\left(\frac{\nu}{vd}\right)^{0\cdot35} + b.$$

In the rough pipe the ratio of friction to v^2 increases with velocity, and Stanton and Pannell suggest, for both rough and smooth pipes, an expression of the form

$$\mathrm{F} = \rho v^2\left\{\mathrm{A}\frac{\nu}{vd} + \mathrm{K} + \mathrm{B}\frac{vd}{\nu}\right\},$$

in which K depends only on the roughness of the pipe. It will be noted that this relationship is similar to the one obtained by Messrs. Bairstow and Booth from experiments on the normal resistance of flat plates (p. 213).

Skin Friction

The resistance to the endwise motion of a thin plane through a fluid is usually termed " skin friction ". Expressing the resistance by $f\mathrm{A}v^n$, where A is the wetted surface, the values of f for various surfaces in water were determined by Mr. Froude.† In these experiments a series of flat boards was suspended vertically from a carriage driven at a uniform speed and was towed through the still water in a large basin. The boards were $\frac{3}{16}$ in. thick, 19 in. deep, and varied in length from 1 ft. to 50 ft. The top edge was submerged to a depth of $1\frac{1}{2}$ in., and the boards were fitted with a cut-water, whose resistance was determined separately.

A short résumé of Mr. Froude's results is given on the following page, these particular figures referring to a velocity of 10 ft.-sec.

Here A refers to varnished surfaces or to the painted surfaces of iron ships, B to surfaces coated with paraffin wax, C to surfaces

* *Proc. Roy. Soc.* A, **91**, 1914, p. 46.

† *British Association Report*, 1872.

Length of surface ..	2 Feet.				8 Feet.				20 Feet.				50 Feet			
Value of n	2·00	1·95	2·16	2·00	1·85	1·94	1·99	2·00	1·85	1·93	1·90	2·00	1·83	—	1·83	2·00
Mean resistance in pounds per square foot ..	0·41	0·38	0·30	0·90	0·325	0·314	0·278	0·625	0·278	0·271	0·262	0·534	0·250	—	0·246	0·488
Resistance per square foot over last foot, in pounds	0·390	0·370	0·297	0·730	0·264	0·260	0·263	0·488	0·240	0·237	0·244	0·465	0·226	—	0·233	0·456
	A	B	C	D	A	B	C	D	A	B	C	D	A	B	C	D

coated with tinfoil, and D to surfaces coated with sand of medium coarseness.

The results show that n decreases down to a certain limit, with an increase in length, but is sensibly independent of the velocity. f decreases with an increase in length, becoming approximately constant when the length is large. Owing to viscous drag, those parts of the surface near the prow communicate motion to the water, so that the relative motion is smaller over the rear part of the surface and its drag per square foot is consequently less, as indicated by a comparison of lines 2 and 3 of the foregoing table.

For the case of air, the most reliable work appears to have been done by Zahm,* who has made observations in an air tunnel 6 ft. square, on smooth boards ranging from 2 to 16 ft. in length, at velocities from 5 to 25 miles per hour. The results are similar to those obtained by Froude in water, in that the resistance per square foot diminishes with the length, and, for smooth surfaces, varies as $v^{1 \cdot 85}$. The following results, corresponding to a velocity of 10 ft.-sec., show that the resistances under similar conditions with short boards are approximately proportional to the densities of the

Length (feet) ..	2	4	8	12	16
Mean resistance, pounds per square foot	0·000524	0·000500	0·000475	0·000477	0·000457

two fluids. Thus for a smooth board (Froude's surface A) the resistance is 790 times as great in water as in air.

For strict comparison, experiments carried out with the same values of $\frac{vl}{\nu}$ should be considered. Thus, taking the ratio of ν for air to ν for water as 13 : 1, a velocity of 10 ft.-sec. with the 4-ft. board in water would correspond to a velocity of 32·5 ft.-sec. with the 16-ft. board in air. Taking the resistance in air as proportional to $v^{1 \cdot 85}$, the resistance per square foot of the 16-ft. board at this speed is

$$R = 0 \cdot 000457 \times (3 \cdot 25)^{1 \cdot 85} = 0 \cdot 00402 \text{ lb.},$$
$$\text{and} \qquad R \div v^2 = 0 \cdot 00000380.$$

The value of $R \div v^2$ for the 4-ft. board in water is 0·00325, the

* *Phil. Mag.*, 8, 1904, pp. 58–66.

ratio of the two being 855, a value only about 4 per cent greater than the relative density of water and air at 60° F.

In view of the fact that one set of experiments was carried out in still water, and the other in an air current whose flow was not perfectly uniform, the agreement between the two sets of results is very close.

Resistance of Wholly Submerged Bodies

Where a body is submerged in a current to such a depth that no surface waves are formed, gravity has no effect on the resistance. This will happen with a deeply submerged submarine, or with an air-ship. If the speed is constant, so that there is no acceleration, and if the liquid is incompressible, or if in a compressible fluid the speeds do not approach the acoustic speed so that pressure changes are so small that the compressibility may be neglected, the resistance R may evidently depend upon the relative velocity of fluid and body, on the density and viscosity of the fluid, and on the size and shape of the body. In a series of geometrically similar bodies each is defined by a single linear dimension l, and the resistance R will be given by the relationship

$$F(R, l, v, \rho, \mu) = 0.$$

As in the previous examples

$$K_1 = R/l^x v^y \rho^z; \quad K_2 = \mu/l^a v^b \rho^c.$$

Inserting the dimensions of l, v, ρ, R, and μ, and determining the indices x, y, z, a, b, c, so as to make K_1 and K_2 dimensionless, gives the values of K_1 and K_2. These are

$$K_1 = \frac{R}{l^2 v^2 \rho}; \quad K_2 = \frac{l\rho v}{\mu}.$$

$$\therefore \quad \psi\left\{\left(\frac{R}{l^2 v^2 \rho}\right), \left(\frac{\mu}{l\rho v}\right)\right\} = 0; \quad \ldots\ldots\ldots\ldots (14)$$

or

$$R = \rho l^2 v^2 \, \phi\left(\frac{l\rho v}{\mu}\right)$$

$$= \rho l^2 v^2 \, \phi\left(\frac{lv}{\nu}\right) \ldots\ldots\ldots\ldots\ldots\ldots (15)$$

The value of the unknown function ϕ might be found by plotting

observed values of $\frac{R}{l^2v^2\rho}$ against simultaneous values of $\frac{lv}{\nu}$, and by finding an empirical equation to represent the curve joining the plotted points. Moreover, it should be noted that since the values of both terms in the function are dependent on v, the form of the function, for any liquid, can be determined from experiments on a single body at different speeds in the same medium.

In the case of model experiments, if the medium is the same for model and original, and if the suffix m denotes the model, then, if the speeds be chosen so that $v_m = \frac{vl}{l_m}$, we shall have

$$\frac{lv}{\nu} = \frac{l_m v_m}{\nu},$$

so that $\phi\left(\frac{lv}{\nu}\right)$ becomes a constant, and

$$\frac{R}{R_m} = \frac{l^2}{l_m{}^2}\frac{v^2}{v_m{}^2} = 1.$$

The speeds thus related are " corresponding speeds ", and at these speeds the model and its original are " dynamically similar ". In this case the corresponding speeds are inversely proportional to the linear dimensions, and at these speeds the resistance of the model and of the original are equal.

Unfortunately this relationship would involve such high speeds in the case of the model as to be of no practical value. If, however, the model experiments can be carried out in a medium whose kinematic viscosity is less than that of the original, the corresponding speeds are reduced. Thus by using compressed air in a wind tunnel the corresponding speed is reduced in the same ratio as the density is increased, since the kinematic viscosity of air varies inversely as its density. Such a wind tunnel is in operation at Langley Field (U.S.A.).

Adopting a working pressure of 20 atmospheres

$$\frac{v_m}{v} = \frac{1}{20}\frac{l}{l_m},$$

so that with a 1/10 scale model, the corresponding speed in the wind tunnel would be one-half that of the original, and at these speeds

$$\frac{R_m}{R} = \frac{\rho_n l^2{}_m v^2{}_m}{\rho l^2 v^2} = \frac{1}{20}.$$

With bodies whose resistance is sensibly proportional to the square of the velocity, the form of the function ϕ must be such as to make $\phi\left(\frac{lv}{\nu}\right)$ a constant whose value depends only on the shape of the body, and the resistance is given by

$$R = k\rho l^2 v^2.$$

It is now immaterial at what speed the model experiments are carried out, so long as this is above the " critical speed ".

This discussion applies equally well to any case of motion of a totally immersed body in a medium whose compressibility may be neglected.

Resistance of Partially Submerged Bodies

When a body is partially submerged, or, although submerged, is so near the surface that surface waves are produced, part of the resistance to motion is due to this wave formation. The influence of gravity must now be taken into account, and we have the relationship

$$F(R, l, v, \rho, \mu, g) = 0.$$

Since there are now 6 quantities involved, $6 - 3\ (= 3)$ K's are required. Taking l, v, and ρ as convenient independent quantities and proceeding as before,

$$K_1 = R/l^x v^y \rho^z,$$
$$K_2 = \mu/l^a v^b \rho^c,$$
$$K_3 = g/l^\alpha v^\beta \rho^\gamma.$$

Inserting the dimensions of R, l, v, ρ, μ, and g, and determining the indices x, a, α, &c., necessary to make K_1, K_2, K_3 dimensionless, gives the values of K_1, K_2, and K_3. These are

$$K_1 = \frac{R}{l^2 v^2 \rho};\ K_2 = \frac{\mu}{lv\rho};\ K_3 = \frac{gl}{v^2};$$

so that
$$\psi\left\{\left(\frac{R}{l^2 v^2 \rho}\right), \left(\frac{\mu}{lv\rho}\right), \left(\frac{gl}{v^2}\right)\right\} = 0,$$

or
$$R = l^2 v^2 \rho\, \phi\left\{\left(\frac{\nu}{lv}\right), \left(\frac{gl}{v^2}\right)\right\}.$$

In the case of model experiments, it is necessary for dynamical similarity that each of the arguments of ϕ shall have the same value for both model and original. But if, as is usual in practice, both are to operate in water, ν is sensibly constant, while g is also constant, so that for exact similarity both lv and v^2/l would require to be constant. In other words, neither v nor l can vary. It follows that the lines of flow and the wave formation around a ship and its model in the same fluid cannot simultaneously be made dynamically similar. It remains to be seen whether any of the arguments involved in the function ϕ may reasonably be neglected so as to give an approximation which is likely to be of use in practice.

Experiment shows that the resistance of a ship-shaped body at the speeds usual in practice is proportional to v^n, where n is approximately 1·83, and the nearness of this index to 2·0 indicates that the direct effect of viscosity is small. If it be assumed that this influence of viscosity is negligible,* the argument ν/lv may be omitted from the equation, which now becomes,

$$\mathrm{R} = l^2 v^2 \rho\, \phi\left(\frac{gl}{v^2}\right).$$

If now $\dfrac{l}{v^2}$ be made the same both for the model and the original,

$$\frac{\mathrm{R}}{\mathrm{R}_m} = \frac{\rho}{\rho_m}\frac{l^2}{l_m^{\,2}}\frac{l}{l_m} = \frac{\rho}{\rho_m}\left(\frac{l}{l_m}\right)^3 = \frac{\rho}{\rho_m}\frac{\mathrm{D}}{\mathrm{D}_m},$$

where D is the displacement.

In this case the " corresponding speeds " at which wave and eddy formation are the same for ship and model, are given by

$$\frac{v}{v_m} = \sqrt{\frac{l}{l_m}}.$$

Model Experiments on Resistance of Ships

In practice these corresponding speeds are used, but allowance is made for the different effects of viscous resistances in the two cases by the well-known " Froude " method.

* This does not involve the assumption that skin friction is unimportant or that viscosity plays no part in the phenomenon. It is in effect assuming that skin friction, instead of being proportional to v^n where n is slightly less than 2, is proportional to v^2. In this case the resistance is due to the steady rate of formation of eddies at the surface of the body, and, once these have been formed and have left the immediate vicinity of the body, the rate at which they are damped out by viscosity has no effect on the drag.

In determining the resistance of any proposed ship, a scale model is made, usually of paraffin wax, and is towed through still water, the resistance corresponding to a number of speeds being measured

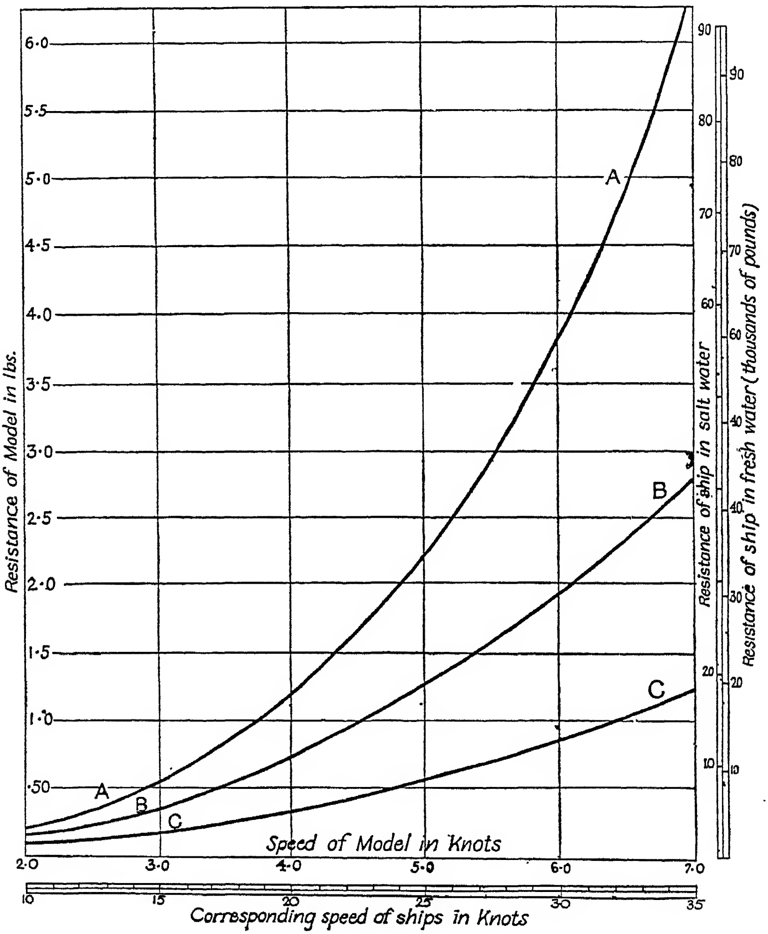

Fig. 3

by dynamometer. A curve AA, fig. 3, is plotted showing resistance against speed.

The length and area of the wetted surface being known, the skin friction ($f_m A_m v_m^n$) is calculated, the coefficient being taken from Froude's results on the resistance of flat planes towed endwise.*

* See pp. 203, 204.

The curve BB of frictional resistance can now be drawn, and the intercept between AA and BB gives the eddy- and wave-making resistance of the model. If now the horizontal scale be increased in the ratio $\sqrt{S : 1}$, where S is the scale ratio of ship and model, and if the vertical scale be increased in the ratio D : 1, this intercept gives the eddy- and wave-making resistance of the ship at the corresponding speed. If fresh water is used in the tank, the vertical scale is to be increased again in the ratio of the densities of salt and fresh water. The skin friction (fAv^n) of the ship is next calculated and set down as an ordinate from BB to give the curve CC. The intercept between the curves AA and CC now gives, on the large scales, the total resistance of the ship.

The resistance at any speed v may be calculated from model observations at the corresponding speed v_m, as follows:

Total resistance of model (observed) .. .. $= R_m$ lb.

Skin friction of model (calculated) .. .. $= f_m A_m v_m^n$ lb.

∴ Wave- + eddy-resistance of model .. $= R_m - f_m A_m v_m^n = F$ lb.

∴ Wave- + eddy-resistance of ship in salt water .. .. $= DF \times \frac{64}{62 \cdot 4}$ lb.

∴ Total resistance of ship .. .. $= \frac{64}{62 \cdot 4} DF + fAv^n$ lb.

Scale Effects—Resistance of Plane Surfaces—of Wires and Cylinders—of Strut Sections

From what has already been said, it will be evident that in most model experiments some one of the factors involved tends to prevent exact similarity and introduces some scale effect. It is only when this effect is small, and when its general result is known, that the results of model experiments can be used with confidence to predict the performance of a large-scale prototype.

The resistance of square plates exposed normally to a current,

affords a typical example of scale effects. Expressing the resistance of such plates as

$$R = K'\rho v^2 \text{ in absolute units (British or C.G.S.)}$$
$$= Kv^2, \text{ where R is in pounds weight per square foot, and } v \text{ is in feet per second,}$$

experiments show the following values of K′ and of K.

	K′	K	Size of Plate.
Dines * ..	·56	·00135	1 ft. square.
Canovetti † ..	·56	·00134	3 ft diameter (circular)
Eiffel ‡ ..	·55	·00133	10 in. square.
,, ...	·56	·00136	14 ,, ,,
,, ...	·59	·00142	20 ,, ,,
,, ...	·61	·00147	27 ,, ,,
,, ...	·62	·00150	39 ,, ,,
Stanton § ..	·52	·00126	2 ,, ,,
,, ...	·62	·00148	5 ft. square
,, ...	·62	·00149	10 ft. square.

Such experiments show that while $\frac{R}{l^2v^2\rho}$ is almost independent of v, it increases by about 18 per cent as the size of the plate is increased from 2 in. to 5 ft. square.

As the compressibility of the air has been neglected in deducing expression (15), p. 206, it is impossible to say without further examination that this effect is not due to compressibility. Indeed if compressibility has any influence on R, a dimensional effect can occur which may be in accordance with a v^2 law of resistance, for, when this is taken into account, (15) becomes

$$R = \rho l^2 v^2 \phi\left(\frac{lv}{\nu}, \frac{v}{C}\right), \ldots\ldots\ldots\ldots\ldots (16)$$

where C is the velocity of sound waves in the medium. This may be written

$$R = \rho l^2 v^2 \phi\left(\frac{lC}{\nu}, \frac{v}{C}\right),$$

* *Proc. Roy. Soc.*, 48, p. 252.

† Société d'Encouragement pour l'Industrie Nationale, *Bulletin*, 1903, **1**, p. 189.

‡ Eiffel, *Résistance de l'Air*.

§ N.P.L., *Collected Researches*, **1**, p. 261.

and since by hypothesis R is proportional to v^2, this becomes

$$R = \rho l^2 v^2 \psi\left(\frac{lC}{\nu}\right).$$

An investigation of the possible effect of compressibility shows, however, that this is less than 1 per cent for speeds up to 100 miles per hour, so that an explanation of the observed dimensional effect based on this factor is not admissible.

It has been suggested* that since, as shown by Mr. Hunsaker's observations on circular discs, there is a critical range of speed determined by the form of the edge, and not dependent on the size of plate, the apparent discrepancy between $\frac{R}{v^2}$ = constant and $\frac{R}{l^2}$ = variable may be due to the results of various observers having been affected by such critical phenomena, which were not, however, sufficiently marked to attract attention. To determine whether this explanation is valid would require an experimental investigation of the forms of edge which have been used.

The more probable explanation would appear to be that while experiments on any one plate have been taken as showing the resistance to be proportional to v^2, this is only approximately true.

An examination by Messrs. Bairstow and Booth,† of all reliable experiments on square plates, leads to the conclusion that an empirical formula of the type

$$R = a(vl)^2 + b(vl)^3 \quad \ldots\ldots\ldots\ldots (17)$$

gives a close approximation to the experimental results. For values of vl ranging from 1 to 350, a and b have the values 0·00126 and 0·0000007 respectively. Here v is in feet per second and R in pounds.

Neglecting compressibility this indicates that $\phi\left(\frac{lv}{\nu}\right)$ of equation (16), p. 212, equals $m + nvl$, where m and n are constants for any particular fluid under given pressure and temperature conditions.

It may be noted that experiments by Stanton‡ show that the pressures on the windward side of a square plate are not subject to a dimensional effect, but that the whole variation can be traced to changes in the negative pressure behind the plate.

* By Mr. E. Buckingham, *Smithsonian Miscellaneous Collections*, **62**, No. 4, Jan., 1916.

† *Technical Report, Advisory Committee for Aeronautics*, 1910–1, p. 21.

‡ *Proc. Inst. C. E.*, **171**; also *Collected Researches of the National Physical Laboratory*, **5**, p. 192.

Resistance of Smooth Wires and Cylinders

A somewhat similar scale effect is obtained from experiments on the resistance of smooth wires and cylinders. A series of such tests on a range of diameters from 0·002 in. to 1·25 in., with v ranging from 10 to 50 ft.-sec.,* shows that, on plotting $\frac{R}{\rho v^2 d^2}$ against $\frac{vd}{\nu}$ or $\log \frac{vd}{\nu}$, a narrow band of points is obtained which in-

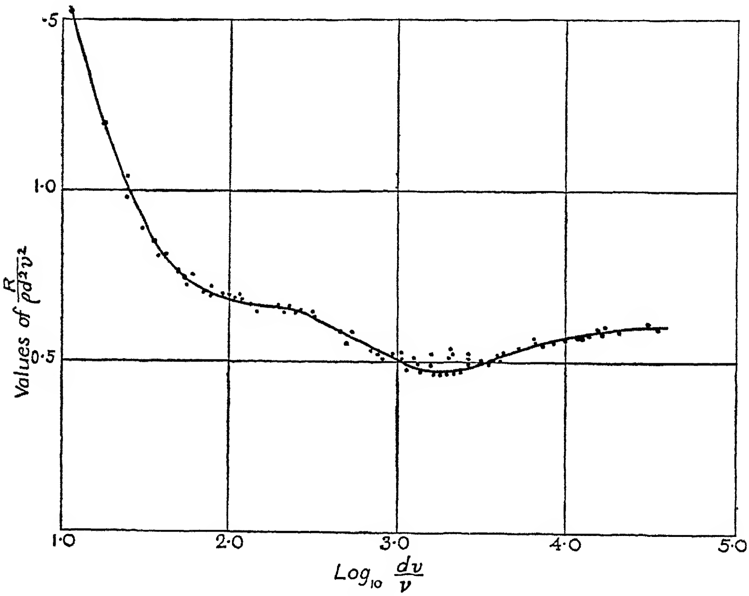

Fig 4

cludes all the experimental results (fig. 4). This shows that for a given value of $\frac{vd}{\nu}$ the value of $\frac{R}{\rho d^2 v^2}$ is the same for all values of v and of d. From this it appears that whereas experiments on a single wire or cylinder indicate that R is nearly proportional to v^2, true similarity of flow is only obtained when $\frac{vd}{\nu}$ is constant.

It becomes very necessary to satisfy this condition with low values of $\frac{vd}{\nu}$, owing to the changes which may occur in the type

* *Reports and Memoranda, Advisory Com. for Aeronautics*, No. 40, March, 1911; No. 74, March, 1913; No. 102.

of flow around such bodies at comparatively low velocities or with small diameters. As in all other cases of flow, as this factor is reduced a critical value is ultimately reached where the type of flow undergoes a definite and rapid change, so that the function ϕ ceases to be even approximately constant.

For a given body in a given medium, this critical value of $\frac{vd}{\nu}$ corresponds to a critical speed which may be calculated from the

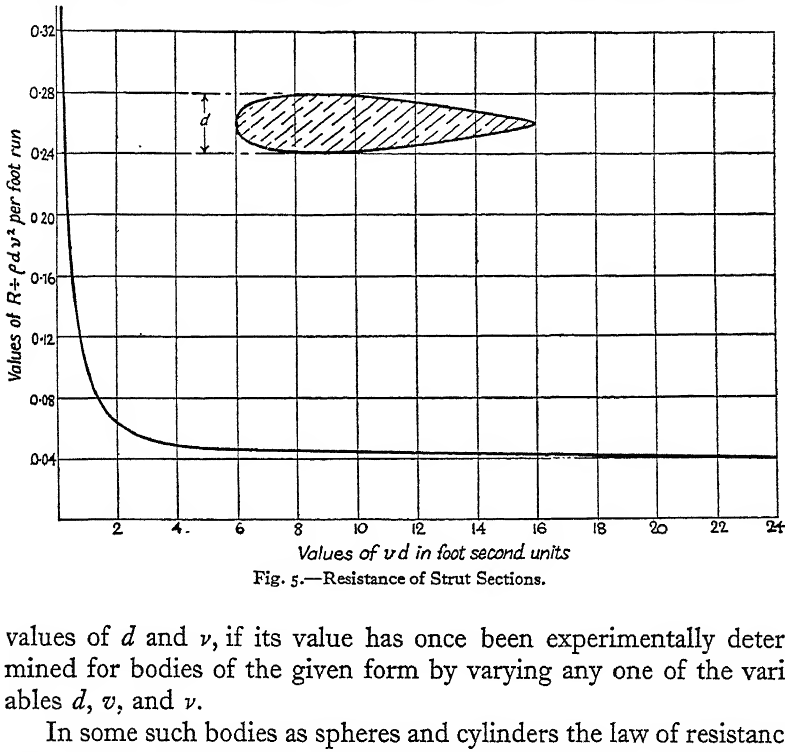

Fig. 5.—Resistance of Strut Sections.

values of d and ν, if its value has once been experimentally determined for bodies of the given form by varying any one of the variables d, v, and ν.

In some such bodies as spheres and cylinders the law of resistance may change widely with comparatively small alteration in the conditions; thus, for example, at certain speeds the resistance of a sphere may actually be reduced by roughening the surface. In carrying out any such experiments, therefore, it is of the greatest importance that the geometrical similarity between a model and its prototype should be as exact as possible, and that where possible vd should be kept constant.

The variation in the type of flow at a definite critical velocity has been well shown in the case of flow past an inclined plate by

C. G. Eden*. By the aid of colour bands in the case of water, and smoke in the case of air, photographs of the eddy formation in the rear of the plates of different sizes were obtained. These show

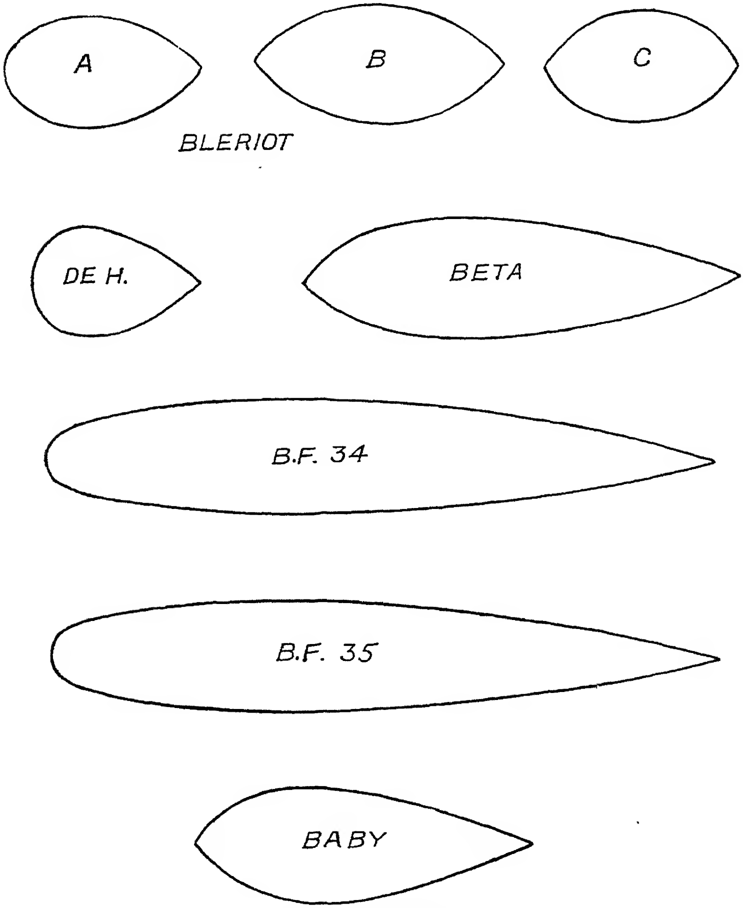

Fig. 6.—Strut Sections.

that the types of flow were similar for both fluids and for all the plates so long as $\frac{vl}{\nu}$ was maintained constant, and that the change over from one type to the other took place at a critical velocity, defined by $v_{\text{crit.}} \propto \frac{\nu}{l}$, in each case.

* *Tech. Report of Advisory Committee for Aeronautics*, 1910–1, p. 48; also *R. and M.*, No. 31, March, 1911.

The curve of fig. 5*, p. 215, shows the change in $\frac{R}{\rho d v^2}$ with a variation in vd in the case of a strut of fair stream-line form. Here R is the resistance per foot run of the strut. The curve shows that the resistance is very nearly proportional to v^2 for values of vd greater than 5, but that as vd is reduced below this value the law of resistance suffers a rapid change. Since, below the critical velocity, $R \propto v$, the ordinates of the curve to the left of the critical point will be proportional to $\frac{1}{v}$, and this part of the curve will be hyperbolic.

The following table† shows the resistance of typical strut sections of the types and sizes shown in fig. 6.

Type of Strut.	Resistance of 100 Ft. of Strut in Pounds at 60 ft.-sec.
Circle, 1 in. diameter ..	43·0
Ellipse axes, 1 in. × 2 in...	22·2
Ellipse axes, 1 in. × 5 in...	15·2
De Havilland	25·5
Farman	22·9
Bleriot A	23·7
Bleriot B	24·5
Baby	7·9
Beta	6·9
B. F. 34	7·2
B. F. 35	6·3
B. F. 35, tail foremost ..	10·9

* *Applied Aerodynamics*, Bairstow (Longmans, Green, & Co., 1920), p. 392.
† *Tech. Report of Advisory Committee for Aeronautics*, 1911–2, p. 96.

CHAPTER VI

Phenomena due to the Elasticity of a Fluid

Compressibility

Compressibility is defined (Chapter I, p. 16) as the reciprocal of the bulk modulus, i.e. by $\frac{1}{v}\left(\frac{\partial v}{\partial p}\right)_T$.

The compressibility of water varies with the temperature and the pressure, the values of the bulk modulus, obtained by different observers, being as follows.* These values are in pounds per square inch.

Authority.	Temperature, Degrees C.						
	0°.	10°.	20°.	30°.	40°.	50°.	
Landholt and Bornstein	284000	303000	318000	333000	—	—	—
Grassi † ..	293000	303000	319000	322000	—	—	—
Tait ‡ ..	283000	301000	319000	334000	347000	352000	At low pressures.
	292000	311000	328000	340000	347000	347000	At 1 ton per sq. in.
	300000	321000	332000	346000	339000	339000	At 2 tons per sq. in.

The bulk modulus K of sea water is about 9 per cent greater than that of fresh water.

* See also Parsons and Cook, *Proc. Roy. Soc.* A, 85, 1911, p. 343. At 4° C. Parsons finds K = 306,000 lb. per square inch at 500 atmospheres; 346,000 lb. at 1000 atmospheres; and 397,000 at 2000 atmospheres. Results of experiments by Hyde, *Proc. R. S.* A, 97, 1920, are in close agreement with these.

† *Annales de Chimie et Physique*, 1851, **31**, p. 437.

‡ *Math. and Phys. Papers*, Sir W. Thomson, Vol. III, 1890, p. 517

For purposes of calculation at temperatures usual in practice, the modulus for fresh water may be taken as 300,000 lb. per square inch, or $43 \cdot 2 \times 10^6$ lb. per square foot.

The compressibility is so small that in questions involving water at rest or in a state of steady flow it may be assumed to be an incompressible fluid. In certain important phenomena, however, where a sudden initiation or stoppage of motion is involved the compressibility becomes an important, and often the predominating factor.

In such cases the true criterion of the compressibility or elasticity of a fluid is measured by the ratio of its bulk modulus to its density, since it is this ratio which governs the wave propagation on which such phenomena depend. In this respect air is only about eighteen times as compressible as water.

For olive oil the value of K at 20° C. is 236,000 lb. per square inch (Quincke) and for petroleum at 16·5° C. is 214,000 lb. per square inch (Martini). The following are values of K for lubricating oils at 40° C.*

Pressure, Tons per square inch.	Castor Oil.	Sperm Oil.	Mobiloil "A".
1	291000	242000	287000
2	302000	252000	291000
5	330000	285000	315000

Sudden Stoppage of Motion—Ideal Case

If a column of liquid, flowing with velocity v along a rigid pipe of uniform diameter and of length l ft., has its motion checked by the instantaneous closure of a rigid valve, the phenomena experienced are due to the elasticity of the column, and are analogous to those obtaining in the case of the longitudinal impact of an elastic bar against a rigid wall.

At the instant of closure the motion of the layer of water in contact with the valve becomes zero, and its kinetic energy is converted into resilience or energy of strain, with a consequent sudden rise in pressure. This checks the adjacent layer, with the result that a state of zero velocity and of high pressure (this at any point

* Hyde, *Proc. R. S.* A, 97, 1920.

being p above the pressure obtaining at that point with steady flow) is propagated as a wave along the pipe with velocity V_p.*

This wave reaches the open end of the pipe after t sec., where

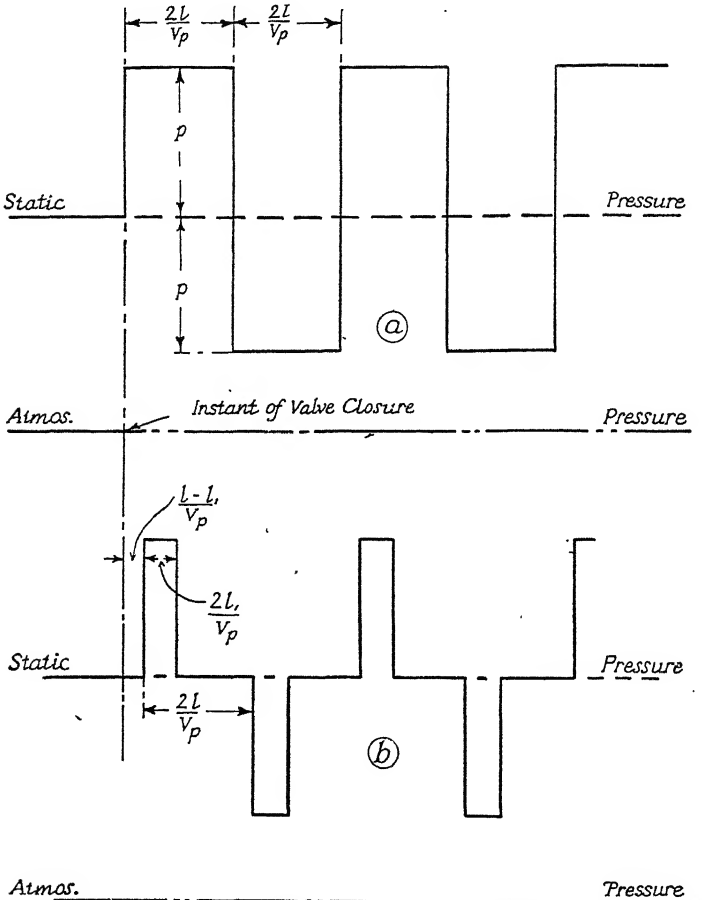

Fig. 1.—Ideal Case of Water-hammer in Pipe Line

$t = l \div V_p$. At this instant the column of fluid is at rest and in a state of uniform compression.

This is not a state of equilibrium, since the pressure immediately

* V_p is the velocity of propagation of sound waves in the medium, and is equal to $\sqrt{\frac{Kg}{w}}$, where w is the weight in pounds per cubic foot, and K is in pounds per square foot.

inside the open end of the pipe is p greater than that in the surrounding medium. In consequence the strain energy of the end layer is reconverted into kinetic energy, its pressure falls to that of the surrounding medium, and it rebounds with its original velocity v towards the open end of the pipe. This relieves the pressure on the adjacent layer, with the result that a state of normal pressure and of velocity $(-v)$ is propagated as a wave towards the valve, reaching it after a second interval $l \div V_p$ sec. At this instant the whole of the column is at normal pressure, and is moving towards the open end with velocity v. The end of the column tends to leave the valve, but cannot do so unless the pressure drops to zero, or so near to zero that any air in solution is liberated. Its motion is consequently checked, and its kinetic energy goes to reduce the strain energy to a value below that corresponding to normal pressure. The pressure drops suddenly by an amount equal to that through which it originally rose, and a wave of zero velocity and of pressure p below normal is transmitted along the pipe, to be reflected from the open end as a wave of normal pressure and velocity towards the valve. When this wave reaches the valve $4l \div V_p$ sec. after the instant of closure, the conditions are the same as at the beginning of the cycle, and the whole is repeated.

Under such ideal conditions the state of affairs at the valve would be represented by fig. 1 *a*. At any other point at a distance l_1 from the open end the pressure-time diagram would appear as in fig. 1 *b*.

If the velocity were such as to make p greater than the normal absolute pressure in the pipe, the first reflux wave would tend to reduce the pressure below zero. Since this is impossible, the pressure could only fall to zero, and the subsequent rise in pressure would be correspondingly reduced. Actually, at such low pressures any dissolved air is liberated and the motion rapidly breaks down.

Effect of Friction in the Pipe Line

The effect of friction in the pipe line modifies the phenomena in a complex manner. In the first place the pressure, with steady flow, falls uniformly from the open end towards the valve, and the pressure at the valve will be represented by such a line as AB (fig. 2). On closure the pressure here will rise by an amount p as before.

When the adjacent layer is checked its rise in pressure will also be p, but since its original pressure was higher than that at the valve, its new pressure will also be higher. It will therefore tend to

compress that portion of the column ahead of it, and will lose some of its strain energy in so doing. This will result in the pressure at the valve increasing as layer after layer is checked, but since this secondary effect travels back from each layer in turn with a velocity V_p, the full effect at the valve will not be felt until a time $2l \div V_p$ after closure. At this instant the pressure will have risen by an amount which to a first approximation may be taken as $\partial p \div \sqrt{2}$,* where ∂p is the pressure-difference at the ends of the pipe under steady flow.

When reflux takes place the end layer, having transmitted part

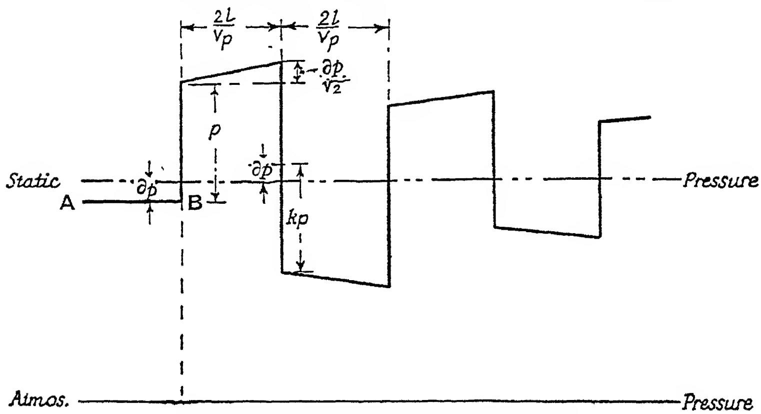

Fig. 2.—Water-hammer as Modified by Friction

of its energy along the pipe, can no longer rebound with the original velocity v. Moreover, since at the instant when the disturbance again reaches the valve, and the column is moving towards the open end, frictional losses necessitate the pressure at the valve being ∂p greater than at the open end, the pressure drop will be less than in the ideal case.

* With steady flow the pressure distant x from the valve will be greater than that at the valve by an amount $\partial p \frac{x}{l}$. Therefore excess strain energy of a layer of length δx at this point, due to this pressure $\propto \partial x \left(\frac{\partial p \times x}{l}\right)^2$.

Assuming this excess energy to be distributed over the length x between this layer and the valve, it will cause a rise in pressure p', where $(p')^2 x = \partial x \left(\frac{\partial p \times x}{l}\right)^2$.

Integrating to obtain the effect of all such layers from 0 to x gives

$$(p')^2 = \left(\frac{\partial p}{l}\right)^2 \int^x x dx = \left(\frac{\partial p}{l}\right)^2 \frac{x^2}{2}, \text{ and when } x = l,\ p = \frac{\partial p}{\sqrt{2}}.$$

This is only a first approximation, since the equalization of pressures will be accompanied by surges which will introduce additional frictional losses.

If the velocity of the first reflux be kv, where k is somewhat less than unity, the pressure diagram will be modified sensibly as in fig. 2.

Friction thus causes the pressure wave to die out rapidly, without affecting the periodicity appreciably.

Magnitude of Rise in Pressure, following Sudden Closure

Assuming a rigid pipe, on equating the loss of kinetic energy per pound of fluid to the increase in its strain energy or resilience:

$$\frac{v^2}{2g} = \frac{p^2}{2Kw}.*$$

$$\therefore\ p = v\sqrt{\frac{Kw}{g}} = vV_p\frac{w}{g},$$

where p is the rise in pressure, and v the velocity of flow in feet per second.

$$\begin{aligned} \text{Putting } K &= 43{\cdot}2 \times 10^6 \text{ lb.-sq. ft.,} \\ \text{,, } \quad w &= 62{\cdot}4 \text{ lb.-c. ft.,} \\ \text{,, } \quad g &= 32{\cdot}2 \text{ ft.-sec.}^2, \\ \text{this becomes } p &= 9160v \text{ lb.-sq. ft.} \\ &= 63{\cdot}7v \text{ lb.-sq. in.} \end{aligned}$$

Effect of Elasticity of Pipe Line

Owing to the elasticity of the pipe walls, part of the kinetic energy of the moving column is expended in stretching these, with a resultant increase in the complexity of the phenomena, a reduction in the maximum pressure attained, and an increase in the rate at which the pressure waves die out. The state of affairs is then indicated in figs. 3 *a* and *b*, which are reproduced from pressure-time diagrams taken from a cast-iron pipe line 3·75 in. diameter and 450 ft. long.†

Fig. 3 *a* was obtained from behind the valve and fig. 3 *b* at a point 15 ft. from the open end of the pipe.

The elasticity of the pipe line may modify the results in two

* If a cube of unit side be subject to a pressure increasing from 0 to p, the change in volume will be $p \div K$, and since the mean pressure during compression is $p \div 2$, the work done is $p^2 \div 2K$.

† Gibson, *Water Hammer in Hydraulic Pipe Lines* (Constable & Co., 1908).

ways. If the pipe is free to stretch longitudinally, at the instant of closure the valve end of the pipe and the water column will move together with a common velocity u,* less than v, and a wave of longitudinal extension will travel along the pipe wall. The instantaneous rise in pressure at the valve will now be equal to

$$(v - u)\sqrt{\frac{Kw}{g}}.$$

Since the velocity of propagation is much greater in metal than

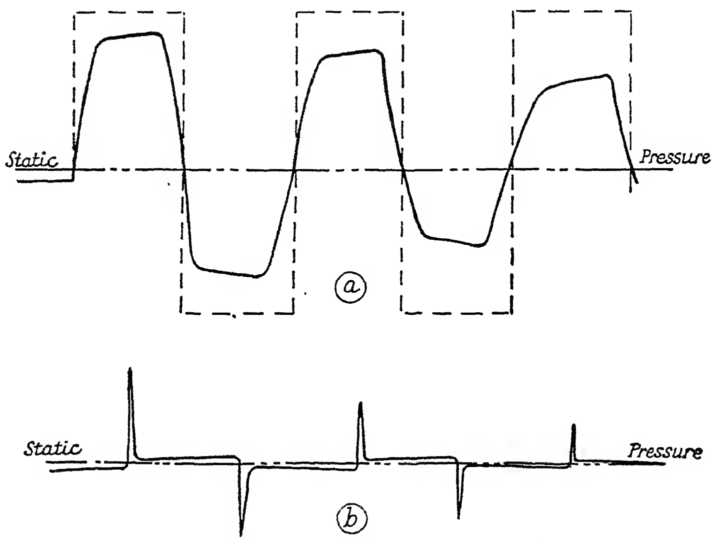

Fig. 3.—Water-hammer in Experimental Pipe Line

in water, this wave will be reflected from the open end of the pipe and will reach the closed end again before the reflected wave in the water column. At this instant the closed end of the pipe will rebound towards the open end with velocity u, and will produce an auxiliary wave of pressure equal to

$$u\sqrt{\frac{Kw}{g}}$$

* It may readily be shown that $u = v\left\{\frac{1}{1 + \frac{w_m a_m V_m}{w a V_p}}\right\}$, where w_m, a_m, and V_m refer to the weight per cubic foot, the cross-sectional area, and the velocity of propagation in the metal, and w, a, V_p, to the corresponding quantities for water.

in the water column. This will increase the pressure to the value

$$v\sqrt{\frac{Kw}{g}},$$

which would obtain if the pipe were anchored. So that the effect on the maximum pressure attained during this first period $2l \div V_p$ is zero. The effect on the subsequent history of the phenomenon is complex. The net effect, however, is to superpose on the normal pressure wave a subsidiary wave of high frequency ($V_m \div 4l$, where V_m is the velocity of propagation of waves in metal) and of magnitude

$$\pm u\sqrt{\frac{Kw}{g}}.$$

If, as is usual in practice, the pipe is anchored so that no appreciable movement of the end is possible, this effect will be small.

The second effect of the elasticity of the pipe line is due to the fact that, since the walls extend both longitudinally and circumferentially under pressure, the apparent diminution of volume of the fluid under a given increment of pressure is greater than in a rigid pipe.

The effect of this is to reduce the virtual value of K to a value K′, where *

$$\frac{1}{K'} = \frac{1}{K} + \frac{r}{2tE}\left(5 - \frac{4}{\sigma}\right),$$

where r is the radius of the pipe, t is the thickness of the pipe walls, E is the modulus of elasticity of the material, $1/\sigma$ is Poisson's ratio for the material (approximately 0·28 for iron or steel).

If the pipe is so anchored that all longitudinal extension is prevented, but that circumferential extension is free, this becomes

$$\frac{1}{K'} = \frac{1}{K} + \frac{2r}{tE}.$$

The rise in pressure due to sudden stoppage of motion is now equal to

$$v\sqrt{\frac{K'w}{g}}.$$

* *Hydraulics and its Applications*, Gibson (Constable & Co., 1912), p. 235.

Valve Shut Suddenly but not Instantaneously

If the time of closure t, while being finite, is so small that

$$t < \frac{l}{V_p} = \frac{x}{V_p},$$

the disturbance initiated at the valve has travelled a distance x, and has not arrived at the open end when the valve reaches its seat. In this case, if each part of the column is subject to the same retardation (a), the relationship

$$\text{force} = \text{mass} \times \text{acceleration}$$

gives $$p = \frac{wax}{g},$$

and since $$a = \frac{v}{t} = \frac{vV_p}{x},$$

this makes $$p = \frac{vV_p w}{g} = v\sqrt{\frac{Kw}{g}} \text{ lb. per square foot,}$$

the value obtained with instantaneous stoppage. Whatever the law of valve closure then, if this is completed in a time less than $l \div V_p$, the pressure rise will be the same as if closure were instantaneous.

Sudden Stoppage of Motion in a Pipe Line of non-Uniform Section

In such a case the phenomena become very complicated. Let l_1, l_2, l_3, &c., be the lengths of successive sections of a rigid pipe, of areas a_1, a_2, a_3. Following sudden closure of a valve at the extremity of the length l_1, a wave of zero velocity and of pressure $63{\cdot}7\,v_1$ lb. per square inch above normal is transmitted to the junction of pipes 1 and 2. Here the pressure changes suddenly to $63{\cdot}7v_2$ above normal. This is maintained in the second pipe during the passage of the wave, and is followed by a change of pressure to $63{\cdot}7v_3$ at the junction of 2 and 3, and so on to the end of the line. But immediately the pressure at the junction of 1 and 2 attains its value $63{\cdot}7v_2$, the wave in pipe 1 is reflected back to the valve as a wave of pressure $63{\cdot}7v_2$ and of velocity $v_1 - v_2$, to be reflected from the valve as a wave of zero velocity and pressure $63{\cdot}7\{v_2-(v_1-v_2)\}$ above normal.

This wave then travels to and fro along pipe 1, making a complete journey in time $l_1 \div V_p$ sec., until such time as the wave in pipe 2,

reflected from the junction of 2 and 3 with pressure $63 \cdot 7 v_3$ above normal and with velocity $v_2 - v_3$, again reaches the junction of 1 and 2. At this instant it takes up a velocity and pressure depending on that at the junction end of pipe 1, and as this depends on the ratio of the lengths of the branches 1 and 2, it is evident that after the first passage of the wave the phenomenon becomes very involved.

Where a pipe is short the period of the oscillations of pressure at any point becomes so small that the pencil of an ordinary indicator is unable to record them, and simply records the mean pressure in the pipe. Thus where a short branch of small diameter is used as the outlet from a long pipe of larger bore, the pressure as recorded by an indicator at the valve will be sensibly the same at any instant as in the large pipe at the point of attachment of the outlet branch.

Moreover, where a non-uniform pipe contains one section of appreciably greater length than the remainder, this will tend to impose its own pressure-change on an indicator placed anywhere in the pipe.

These points are illustrated by the following results of experiments by S. B. Weston.* In each case the outlet valve was on the 1 in. length.

Details of Pipe Line.	Pressures, Pounds per Square Inch.					
	Calc.	Obs.	Calc.	Obs.	Calc.	Obs.
111 ft. of 6-in. pipe.						
58 „ 2 „	154	73	69	71	—	—
99 „ 1½ „	322	129	143	127	8·9	14·5
4 „ 1 „						
111 ft. of 6-in. pipe.	1st 1½-in. pipe.		3-in. pipe.		2nd 1½-in. pipe.	
58 „ 2 „						
48 „ 1½ „	71·5	75	18·0	65	71·5	61
3 „ 3 „	142	126	35·5	121	143	114
48 „ 1½ „	180	150	45	150	180	139
4 „ 1 „	268	203	67	207	268	196
182 ft. of 6-in. pipe.	1½-in. pipe.		2½-in. pipe.		6-in. pipe.	
66 „ 4 „						
4 „ 2½ „	120	49	52	22	9·0	4·8
1 „ 2 „	149	62	64·5	36	11·2	6·6
7 „ 1½ „	223	82	97	52	16·7	15·8
6 „ 1 „	466	122	201	99	35·0	36·8

* *Am. Soc. C. E.*, 19th Nov., 1884.

Sudden Initiation of Motion

If the valve at the lower end of a pipe line be suddenly opened, the pressure behind the valve falls by an amount p lb. per square inch, and a wave of velocity v towards the valve

$$\left(v = p\sqrt{\frac{g}{Kw}} \text{ approx.}\right)$$

and of pressure p below statical pressure is propagated towards the pipe inlet.

The magnitude of p depends on the speed and amount of opening of the valve, and if the latter could be thrown wide open instantaneously the pressure would fall to that obtaining on the discharge side. In experiments by the writer* with a $2\frac{1}{2}$-in. globe valve on a $3\frac{3}{4}$-in. main 450 ft. long, with the valve thrown open through 0·5 of a complete turn, the drop in pressure was 40 lb. per square inch, the statical pressure in the pipe being 45 lb. per square inch, and on the discharge side zero. With the valve opened through 0·10 of a turn the drop was 20 lb. per square inch, while with 0·05 of a turn it was 11 lb. per square inch. In each case the time of opening was less than 0·13 sec. ($l \div V_p$).

With a pipe so situated that the original statical pressure is everywhere greater than p, this pressure wave reaches the pipe inlet with approximately its original amplitude, and at this instant the column is moving towards the valve with velocity v and pressure p below normal.

The pressure surrounding the inlet is however maintained normal, so that the wave returns from this end with normal pressure and with velocity $2v$ relative to the pipe. At the valve the wave is reflected, wholly or in part, with a velocity which is the difference between $2v$ and the velocity of efflux at that instant, and since the velocity of efflux will now be greater than v, the wave velocity will be less than v, and the rise in pressure less than p above normal. This wave is reflected from the inlet to the valve and here the cycle is repeated, the amplitude of the pressure wave diminishing rapidly until steady flow ensues. Fig. 4 shows a diagram obtained under these circumstances.

Where the gradient of the pipe is such that beyond a certain point in its length the absolute statical pressure is less than the drop in pressure at the valve, the motion becomes partly discontinuous

* Gibson, *Water Hammer in Hydraulic Pipe Lines* (Constable & Co., 1908).

at this point on the passage of the first wave, which travels on to the inlet with gradually diminishing amplitude. The amplitude with which it reaches the inlet determines the velocity of the reflected

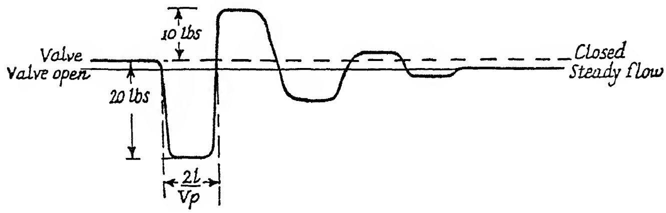

Fig. 4.—Diagram of Pressure (per square inch) obtained on Sudden Opening of a Valve

wave, which will be less than in the preceding case, and under such circumstances the wave motion dies out rapidly.

As the valve opening becomes greater, the efficiency of the valve as a reflecting surface becomes less, so that with a moderate opening the pressure may never even attain that due to the statical head.

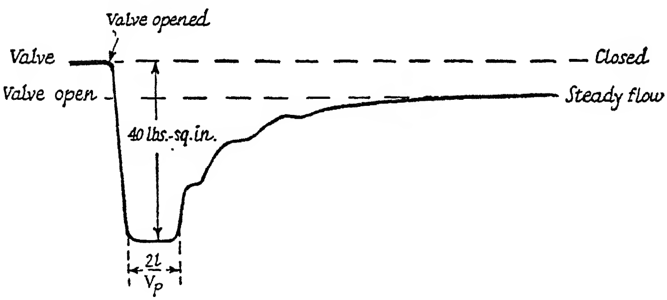

Fig. 5.—Sudden Opening of Valve

This is shown in fig. 5, which is a diagram obtained from the experimental pipe line when the valve was opened suddenly $\left(\text{time} < \frac{l}{V_p}\right)$ through half a turn.

Wave Transmission of Energy

In the systems in common use for the hydraulic transmission of energy, water under a pressure of about 1000 lb. per square inch is supplied from a pumping-station and is transmitted through pipe lines to the motor. This method involves a continuous flow of the working fluid, which in effect serves the purpose of a flexible coupling between the pump and the motor.

It is, however, possible to supply energy to a column of fluid enclosed in a pipe line, to transmit this in the form of longitudinal vibrations through the column, and to utilize it to perform mechanical work at some remote point. Such transmission is possible in virtue of the elasticity of the column.*

If one end of a closed pipe line full of water under a mean pressure p_m be fitted with a reciprocating plunger, a wave of alternate compression and rarefaction is produced, which is propagated along the pipe with velocity V_p. If the plunger has simple harmonic motion, the state of affairs in a pipe line so long that, at the given instant, the disturbance has not had time to be reflected from its further end, is represented in fig. 6. The pressure at each point

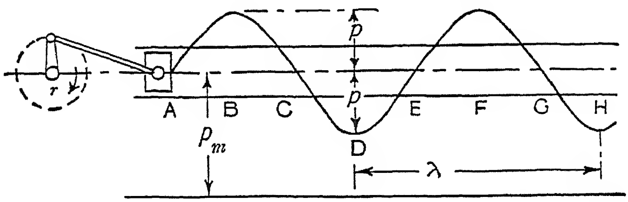

Fig. 6

will oscillate between the values $p_m \pm p$, and the velocity between the values $\pm v$, where v is the maximum velocity of the piston. At the instant in question, particles at A, C, E, and G are oscillating to and fro along the axis of the pipe through a distance r on each side of their mean position, while particles at B, D, F, and H are at rest. If n be the number of revolutions of the crank per second, the wave-length $\lambda = V_p \div n$ ft.

In a pipe closed at both ends such a state of vibration is reflected from end to end, forming a series of waves of pressure and velocity whose distribution, at any instant, depends on the ratio of the length l of the pipe to λ.

In the cases where l is respectively equal to $\lambda/4$, $\lambda/2$, and λ, stationary waves are produced as indicated in fig. 7.‡ The excess

* A number of applications of this method have been patented by Mr. G. Constantinesco.

‡ The pressure and velocity oscillate in a "stationary" manner, i.e. there are definite points called "nodes" where there is no change in pressure and likewise points where the water does not move. See any textbook on Sound, e.g. Datta's *Sound* (Blackie), p. 63, Watson's *Physics*, Poynting and Thomson's *Sound*, &c., where the subject is fully explained for sound waves.

This distribution, where $l = \lambda/4$, is only possible where oscillation at the end A is possible, as where the pipe is fitted with a free plunger.

pressure at a given point oscillates between equal positive and negative values, the range of pressure being given by the intercept between the two curves. The velocity at the points of maximum and minimum pressure, as at A, D, and B in fig. 7 *c*, is zero, while at the points C and E, where the variation of pressure is zero, the velocity varies from $+v$ to $-v$.

In the case where $l = \lambda/4$, the plunger, if free, would continue

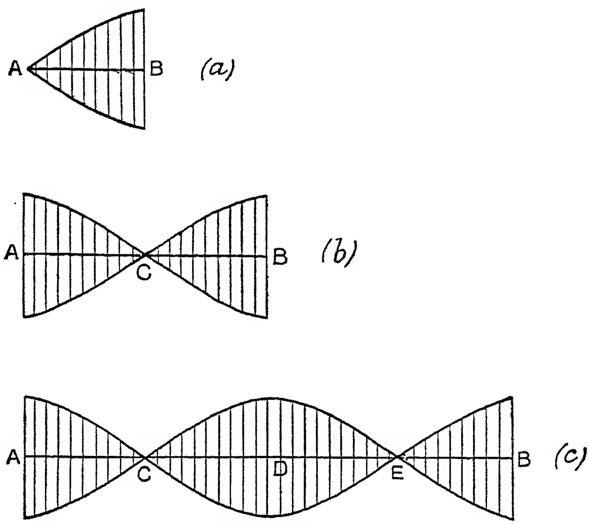

Fig. 7.—Stationary Waves in a Closed Pipe.

to oscillate in contact with the end of the column without the application of any external force.

In a pipe fitted with a reciprocating plunger at one end and closed at the other, the wave system initiated by the plunger will be superposed on this reflected system. Thus if $l = \lambda/2$ or λ, the wave initiated by the plunger will be reflected, and will reach the plunger as a zone of maximum pressure at the instant the latter is completing its in-stroke and is producing a new state of maximum pressure. The pressure due to the reflected wave is superposed on that due to the direct compression, with the result that the pressure is doubled. The next revolution will again increase the pressure, and so on until the pipe either bursts or until the rate of dissipation of energy due to friction, and to the imperfect elasticity of the pipe walls, becomes equal to the rate of input of energy by the plunger.

On the other hand, if the length of the pipe be any odd multiple of $\lambda/4$, the pressure at the plunger at any instant, due to the reflected wave, will be equal in magnitude but opposite in sign to that primarily due to the displacement of the plunger, and the pressure on the latter will be constant and equal to the mean pressure in the pipe. Except for the effect of losses in the pipe walls and in the fluid column, reciprocation may now be maintained indefinitely without the expenditure of any further energy. For any intermediate lengths of pipe, the conditions will also be intermediate and the wave distribution complex.

Instead of closing the end of the pipe at B (fig. 8), suppose a piston to be fitted to a crank rotating at the same angular velocity, in the same direction, and in the same phase as the crank at A. If the column were continued beyond B, the movement of the

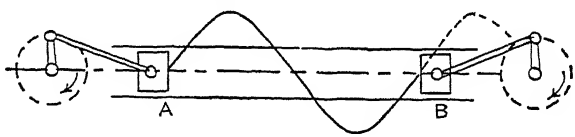

Fig. 8

piston would evidently produce in the column a series of waves forming an exact continuation of the wave system between A and B. There will now be no reflection from the surface of the piston, and if the latter drives its crank and if the resistance is, at every instant, equal to the force exerted on the piston by the wave system, it will take up the whole energy of the waves produced by piston A. It is to be noted that the piston B may be placed at any point in the pipe so long as its phase is the same as that of the liquid at the point of connection.

If more energy is put in by the piston A than is absorbed by B, reflected waves will be formed, and the continuation of the motion will accumulate energy in the system, increasing the maximum pressure until, as in the case of the closed pipe, the pipe will burst.

This may be avoided by fitting a closed vessel, filled with liquid, having a volume large in comparison with the displacement of the piston, in communication with the pipe near to the piston. Alternatively this may be replaced by a spring-loaded plunger. In either case the contrivance acts as a reservoir of energy. If the piston B is not absorbing the whole of the energy supplied from A, the liquid in this chamber is compressed on each instroke of the piston A,

to re-expand on the outstroke, and by giving to it a suitable volume, the maximum pressures, even when the piston B is stationary, may be reduced to any required limits. If perfectly elastic, the reservoir will return as much energy during expansion as it absorbed during compression, so that the net input to the driving piston is only equivalent to that absorbed by piston B.

In the case of a pipe (fig. 7 *c*) whose length is one wave-length, and which is provided with branches at C, D, E, and B, respectively one-quarter, one-half, three-quarters, and one wave-length from A, if all the branches are closed, stationary waves will be produced in the pipe as previously described.

If now a motor running at the synchronous speed be coupled to the branch at D, this will be able to take up all the energy given to the column. The stationary half-wave between A and D will vanish, being replaced by a wave of motion, while the stationary wave will still persist between D and B.

Since there is no pressure variation at C and E, motors coupled at these points, with the remaining branches closed, can develop no energy.

If a motor be connected at any intermediate point, part of the input of energy can be taken up by the motor. The stationary wave will then persist, but be of reduced amplitude between A and the motor, the wave motion over this region being compounded of this stationary wave and of a travelling wave conveying energy.

With a motor at the end B of the line, not absorbing all the energy given by the generator A, there is, in the pipe, a system of stationary waves superposed on a system of waves travelling along the pipe, so that there will be no point in the pipe at which the variation of pressure is always zero. It follows that under these conditions a motor connected at any point of the pipe will be able to take some energy and to do useful work.

In practice a three-phase system is usually employed, as giving more uniform torque and ease of starting. A three-cylinder generator, having cranks at 120°, gives vibrations to the fluid in three pipes, which are received by the pistons of a three-cylinder hydraulic motor having the same crank angles. The mean pressure within the system is maintained by a pump, which returns any fluid leaking past the pistons.

Theory of Wave Transmission of Energy

The simple theory of the process is outlined below, on the assumption that the friction loss due to the oscillation of the column in the pipe is directly proportional to the velocity. Where such a viscous fluid as oil is used this is true, but where water is used it may or may not be true, depending upon the velocities involved. If the resistance is equal to kv^2 per unit length as with turbulent motion, an approximation to the true result may be attained by choosing such a frictional coefficient k' as will make $k'v = kv^2$ at the mean velocity. At velocities below the critical, $k' = \frac{32\mu}{d^2}$ in pounds per square foot of the cross section (Poiseuille) per unit length of the pipe.

Consider the fluid normally in a plane at x, displaced from that plane through a small distance u, so that its velocity $v = \frac{\partial u}{\partial t}$. The difference of pressure on the ends of an element of length δx, due to the variation in compression along the axis of the pipe, is equal to

$$\mathrm{K}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)\delta x,$$

and the equation of motion becomes

$$\rho\frac{\partial^2 u}{\partial t^2}\frac{\pi d^2}{4}\delta x = \mathrm{K}\frac{\partial^2 u}{\partial x^2}\frac{\pi d^2}{4}\delta x - \frac{32\mu}{d^2}\frac{\pi d^2}{4}\frac{\partial u}{\partial t}\delta x,$$

$$\text{or} \quad \frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2} - b\frac{\partial u}{\partial t}, \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$\text{where} \quad a = \sqrt{\frac{\mathrm{K}}{\rho}} \text{ and } b = \frac{32\mu}{\rho d^2}.$$

If b is small compared with $4\pi n$, where n is the frequency of the vibration, a solution of equation (1) is

$$u = u_0 e^{-\frac{1}{2}\frac{b}{a}x}\sin 2\pi n\left(t - \frac{x}{a}\right), \ldots\ldots\ldots\ldots(2)$$

which represents an axial vibration throughout the column, of maximum amplitude u_0 at the end where $x = 0$.

At any other point the maximum amplitude is $u_0 e^{-\frac{1}{2}\frac{b}{a}x}$, gradu-

ally diminishing along the pipe owing to the friction term represented by the term b.

The excess pressure p, at any instant and at any point, is equal to $-K\frac{du}{dx}$, i.e.

$$p = -Ku_0e^{-\frac{1}{2}\frac{b}{a}x} \times \frac{2\pi n}{a}\cos 2\pi n\left(t - \frac{x}{a}\right) \text{ approx.},$$

and the maximum excess pressure, $p_{\text{max.}}$, at any point,

$$= -\frac{2\pi nKu_0}{a}e^{-\frac{1}{2}\frac{b}{a}x}. \quad\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

The velocity of a particle at x is equal to

$$\frac{\partial u}{\partial t} = 2\pi nu_0e^{-\frac{1}{2}\frac{b}{a}x}\cos 2\pi n\left(t - \frac{x}{a}\right),$$

and the maximum velocity,

$$v_{\text{max.}} = 2\pi nu_0e^{-\frac{1}{2}\frac{b}{a}x} \quad\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

The energy transmitted by the excess pressure—the mean pressure conveys no energy on the average—across a given section of the pipe in time δt is equal to

$$\frac{\pi d^2}{4}pv\delta t = -\frac{\pi d^2}{4}K\frac{\partial u}{\partial x}\frac{\partial u}{\partial t}\delta t.$$

The mean rate of transmission of energy per second over each stroke of the plunger is thus given by

$$-\frac{\pi d^2}{4}\frac{K}{\tau}\int_0^\tau \frac{\partial u}{\partial x}\frac{\partial u}{\partial t}dt,$$

where τ is the duration of a stroke, i.e. of a half-cycle.

Writing $2\pi n\left(t - \frac{x}{a}\right) = \alpha$, $dt = \frac{d\alpha}{2\pi n}$, $\tau = \frac{\pi}{2\pi n} = \frac{1}{2n}$,

this becomes $$-\frac{Kd^2}{4}\int_0^\pi \frac{\partial u}{\partial x}\frac{\partial u}{\partial t}d\alpha$$

$$= \frac{\pi K}{2a}(\pi ndu_0)^2e^{-\frac{b}{a}x} \quad\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

$$= \frac{1}{2}p_{\text{max.}}v_{\text{max.}}\frac{\pi d^2}{4} = E,$$

and the horse-power transmitted, if the foot be the unit of length, is obtained by dividing expression (5) by 550.

The loss of energy per unit length of the pipe, due to friction, and converted into heat, is

$$-\frac{\partial E}{\partial x} = \frac{b}{a}E,$$

and the efficiency of transmission through a pipe line of length l is

$$E_l \div E_0 = e^{-\frac{bl}{a}}.$$

It should be noted that in any application of these results, if the calculations are in English units,

$$\rho = \frac{w}{g} = \frac{62{\cdot}4}{32{\cdot}2}$$

for water, while the value of μ is to be taken in pounds per square foot, and the pipe diameter in feet.

For a more detailed investigation of the theory, which becomes complex when a complicated pipe system is used, Mr. Constantinesco's original papers should be consulted*.

There is an exceedingly close analogy between wave transmission by Constantinesco's system and alternating-current electric power transmission; in fact, in the "three-pipe system" the known facts of three-phase electrical engineering can be applied with scarcely any except verbal changes.

* *The Theory of Sonics* (The Proprietors of Patents Controlling Wave Transmission, 132 Salisbury Square, E.C., 1920).

NOTE.—The foregoing theory of wave transmission is due to H. Moss, D.Sc. See also *Proc. Inst. Mech. Eng.*, 1923.

CHAPTER VII

The Determination of Stresses by Means of Soap Films

When a straight bar of uniform cross section is twisted by the application of equal and opposite couples applied at its two ends, it twists in such a way that any two sections which are separated by the same distance are rotated relative to one another through the same angle. The angle through which sections separated by a unit length of the bar are twisted relatively to one another is called the " twist ", and it will be denoted by the symbol $\mathfrak{T}$ throughout this chapter. If the section is circular, particles of the bar which originally lay in a plane perpendicular to the axis continue to do so after the couple has been applied.

The couple is transmitted through the bar by means of the shearing force exerted by each plane-section on its neighbour. The shearing stress at any point is, in elastic materials, proportional to the shear strain, or shear. In the case where a bar of circular cross section is given a twist $\mathfrak{T}$, the shear evidently increases from zero at the axis to a maximum at the outer surface of the bar; at a distance r from the axis it is in fact $r\mathfrak{T}$. If two series of lines had been drawn on the surface of the untwisted bar so as to be parallel and perpendicular to the axis, these lines would have cut one another at right angles. After the twist, however, these lines cut at an angle which differs from a right angle by the angle $r\mathfrak{T}$, which measures the shear at the point in question. The shearing strain at the surface of any twisted bar can in fact be conceived as the difference between a right angle and the angle between lines of particles which were originally parallel and perpendicular to the axis.

In the case of bars whose sections are not circular, the particles which originally lay in a plane perpendicular to the axis do not continue to do so after the twisting couple has been applied; the cross sections are warped in such a way that the shear is increased

in some parts and decreased in others. In the case of a bar of elliptic section, for instance, the point on the surface of the bar where the shear is a maximum is at the end of the minor axis, while the point where it is a minimum is at the end of the major axis. If the sections had remained plane, so that the shear at any point was proportional to the distance of that point from the axis of twist, the reverse would have been the case.

The warping of sections which were originally plane is of fundamental importance in discussing the distribution of stress in bent or twisted bars. It may give rise to very large increases in stress. In the case where the section has a sharp internal corner, for instance, it gives rise to a stress there which is, theoretically, infinitely great.

The method which has been used to discuss mathematically the effect of this warping is due to St. Venant.* If co-ordinate axes Ox, Oy be chosen in a plane perpendicular to the axis of the bar, and if ϕ represents the displacement of a particle from this plane owing to the warping which occurs when the bar is twisted, then St. Venant showed that ϕ satisfies the equation

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2} = 0, \qquad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

and that it must also satisfy the boundary condition

$$\frac{\partial\phi}{\partial n} = y\cos(xn) - x\cos(yn), \qquad \ldots\ldots\ldots\ldots(2)$$

where $\frac{\partial\phi}{\partial n}$ represents the rate of change of ϕ in a direction perpendicular to the boundary of the section, and (xn), (yn) represent the angles between the axes of x and y respectively and the normal to the boundary at the point (x, y).

Functions which satisfy equation (1) always occur in pairs. If ψ is the function conjugate to ϕ, i.e. the other member of the pair, ψ is related to ϕ by the equations

$$\frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y},\quad \frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x}, \qquad \ldots\ldots\ldots\ldots(3)$$

and ψ also satisfies (1). In the case under consideration it turns out that it is simpler to determine ψ and then to deduce ϕ than to attempt to determine ϕ directly. From (1), (2), and (3) it will be seen that to determine ψ it is necessary to find a function ψ which satisfies

* See Love, *Mathematical Theory of Elasticity*, second edition, Chap. XIV.

(1) at all points of the cross section, and satisfies the equation

$$\frac{\partial\psi}{\partial s} = y\cos(xn) - x\cos(yn)\ldots\ldots\ldots\ldots(4)$$

at points on the boundary, where $\frac{\partial\psi}{\partial s}$ represents the rate of variation of ψ round the boundary.

Now $\cos(xn) = \frac{\partial y}{\partial s}$ and $\cos(yn) = -\frac{\partial x}{\partial s}$, so that (4) reduces to $\frac{\partial\psi}{\partial s} = \frac{1}{2}\frac{\partial}{\partial s}\left(x^2+y^2\right)$, that is to say the boundary condition reduces to

$$\psi = \tfrac{1}{2}(x^2+y^2) + \text{constant.}\ldots\ldots\ldots\ldots(5)$$

The advantage in using ψ instead of ϕ is that the boundary condition (5) is more easy to satisfy than (2).

The problem of the torsion of the bar of any section is therefore reduced to that of finding a function ψ which satisfies $\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2} = 0$ and (5). There is an alternative, however. If a function Ψ be defined by the relation $\Psi = \psi - \frac{1}{2}(x^2+y^2)$, then Ψ evidently must satisfy the equation

$$\frac{\partial^2\Psi}{\partial x^2}+\frac{\partial^2\Psi}{\partial y^2} + 2 = 0,\ldots\ldots\ldots\ldots(6)$$

at all points of the section, and

$$\Psi = \text{constant}\ldots\ldots\ldots\ldots(7)$$

at the boundary.

This function Ψ, besides having a very simple boundary condition, has also the advantage that it is simply related to the shear, in fact the shearing strain at any point is proportional to the rate of change in Ψ at the point in question in the direction in which it is a maximum.

Prandtl's Analogy

It has only been possible to obtain mathematical expressions for ϕ, ψ, and Ψ in very few cases. The stresses in bars whose sections are rectangles, ellipses, equilateral triangles, and a few other special shapes have been found, but these special shapes are of little interest to engineers. There is no general way in which the stresses in twisted bars of any section can be reduced to mathematical terms.

The usefulness of equations (6) and (7) does not cease, however when Ψ cannot be represented by a mathematical expression. It has

been pointed out by various writers that certain other physical phenomena can be represented by the same equations. In some cases these phenomena can be measured experimentally far more easily than direct measurements of the stresses and strains in a twisted bar can be made. Under these circumstances it may be useful to devise experiments in which these phenomena are measured in such a way that Ψ is evaluated at all points of the section. The values thus found for Ψ can then be used to determine the stresses in a twisted bar.

Probably the most useful of these " analogies " is that of Prandtl.

Consider the equations which represent the surface of a soap film stretched over a hole in a flat plate of the same size and shape as the cross section of the twisted bar, the film being slightly displaced from the plane of the plate by a small pressure p.

If γ be the surface tension of the soap solution, the equation of the surface of the film is

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} + \frac{p}{2\gamma} = 0, \quad \ldots\ldots\ldots\ldots\ldots (8)$$

where z is the displacement of the film from the plane of xy and x and y are the same co-ordinates as before. Round the boundary, i.e. the edge of the hole, $z = 0$.

It will be seen that if z is measured on such a scale that $\Psi = 4\gamma z/p$, then equations (6) and (8) are identical. The boundary conditions are also the same. It appears therefore that the value of Ψ, corresponding with any values of x and y, can be found by measuring the quantities p/γ and z on the soap film.

In other words the soap film is a graphical representation of the function Ψ for the given cross section. Actual values of Ψ can be obtained from it by multiplying the ordinates by $4\gamma/p$.

To complete the analogy it is necessary to bring out the direct connection between the measurable quantities connected with the film and the elastic properties of the twisted bar.

If N is the modulus of rigidity of the material and $\mathfrak{T}$ the twist of the bar, the shear stress at any point of the cross section can be found by multiplying the slope of the Ψ surface at the point by $\mathrm{N}\mathfrak{T}$, so that, if α is the inclination of the bubble to the plane of the plate, the stress is

$$f_s = \frac{4\gamma}{p}\mathrm{N}\mathfrak{T}\alpha \quad \ldots\ldots\ldots\ldots\ldots\ldots (9)$$

The torque T_q on the bar is given by

$$T_q = 2N\mathfrak{T}\iint \Psi dx,\, dy,$$

$$\text{or} \qquad T_q = \frac{8\gamma}{p} N\mathfrak{T}V, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (10)$$

where V is the volume enclosed between the film surface and the plane of the plate.

The contour lines of the soap film in planes parallel to the plate correspond to the " lines of shearing stress " in the twisted bar, that is, they run parallel to the direction of the maximum shear stress at every point of the section.

It is evident that the torque and stresses in a twisted bar of any section whatever may be obtained by measuring soap films in these respects.

In order to obtain quantitative results, it is necessary to find the value of $\frac{4\gamma}{p}$ in each experiment. This might be done by measuring γ and p directly, but a much simpler plan is to determine the curvature of a film, made with the same soap solution, stretched over a circular hole and subjected to the same pressure difference, p, between its two surfaces as the test film.

The curvature of the circular film may be measured by observing the maximum inclination of the film to the plane of its boundary.

If this angle be called β then

$$\frac{4\gamma}{p} = \frac{h}{\sin\beta}, \quad \ldots\ldots\ldots\ldots\ldots\ldots (11)$$

where h is the radius of the circular boundary.

The most convenient way of ensuring that the two films shall be under the same pressure, is to make the circular hole in the same plate as the experimental hole.

It is evident that, since the value of $4\gamma/p$ for two films is the same, we may, by comparing inclinations at any desired points, find the ratio of the stresses at the corresponding points of the cross section of the bar under investigation to the stresses in a circular shaft of radius h under the same twist. Equally, we can find the ratio of the torques on the two bars by comparing the displaced volumes of the soap films. This is, in fact, the form which the investigations usually take.

As a matter of fact, the value of $\frac{4\gamma}{p}$ can be found from the test-film itself by integrating round the boundary, α, its inclination to the plane of the plate. If A be the area of the cross section, then the equilibrium of the film requires that

$$\int 2\gamma \sin\alpha ds = pA. \dots\dots\dots\dots\dots (12)$$

This equation may be written in the form

$$\frac{4\gamma}{p} = 2 \times \frac{\text{area of cross section}}{(\text{perimeter of cross section}) \times (\text{mean value of } \sin\alpha)}. \quad (13)$$

By measuring α all round the boundary the mean value of $\sin\alpha$ can be found, and hence $\frac{4\gamma}{p}$ may be determined. This is, however, more laborious in practice than the use of the circular standard.

It is evident that if the radius of the circular hole be made equal to the value of $\frac{2A}{P}$, where A is the area and P the perimeter of the test hole, then $\sin\beta$ = mean value of $\sin\alpha$. It is convenient to choose the radius of the circular hole so that it satisfies this condition, in order that the quantities measured on the two films may be of the same order of magnitude.

Experimental Methods

It is seen from the mathematical discussion given above that, in order that full advantage may be taken of the information on torsion which soap films are capable of furnishing, apparatus is required with which three kinds of measurements can be made, namely:

(*a*) Measurements of the inclination of the film to the plane of the plate at any point, for the determination of stresses.

(*b*) Determination of the contour lines of the film.

(*c*) Comparison of the displaced volumes of the test film and circular standard for finding the corresponding torque ratio.

The earliest apparatus designed by Dr. A. A. Griffith and G. I. Taylor for making these measurements is shown in fig. 1 (see Plate).* The films are formed on holes cut in flat aluminium plates

* From *Proc. Inst. Mech. Eng.*, 14th December, 1917.

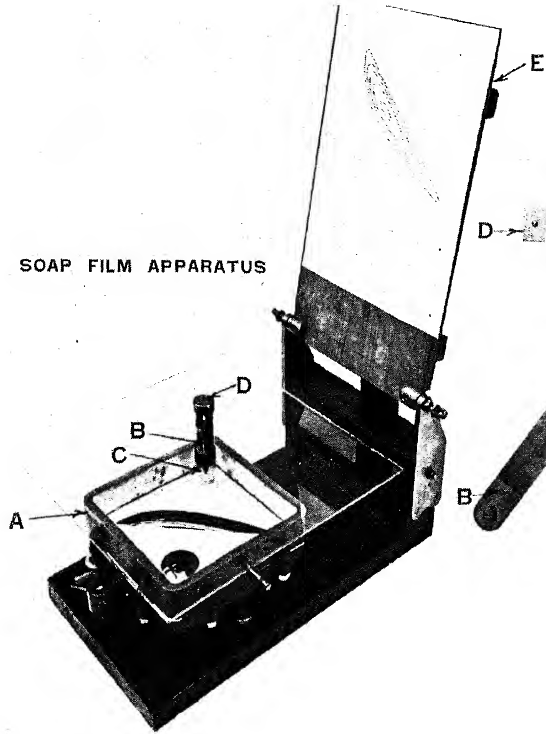

FIG. 1

AUTO-COLLIMATOR

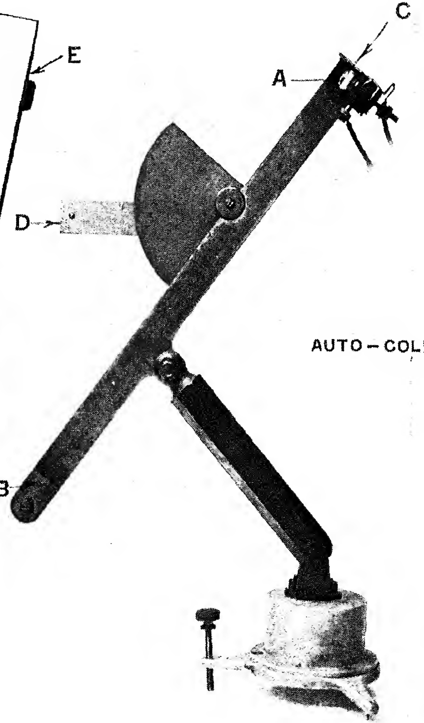

FIG. 2(a)

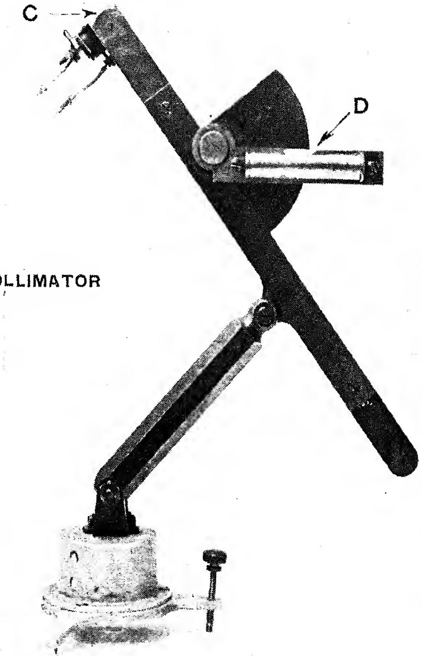

FIG. 2(b)

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to the required shape. These plates are clamped between two halves of the cast-iron box A (fig. 1). The lower part of the box takes the form of a shallow tray $\frac{1}{4}$ in. deep blackened inside and supported on levelling screws, while the upper portion is simply a square frame, the upper and lower surfaces of which are machined parallel. Arrangements are made so that air can be blown into the lower part of the box in order to establish a difference in pressure between the two sides of the film.

In order to map out the contour lines of the film, i.e. lines of equal z, or lines of equal Ψ in the twisted bar, a steel point wetted with soap solution is moved parallel to the plane of the hole till it just touches the film. The point being at a known distance from the plane of the hole marks a point on the film where z has the known value. The required motion is attained by fixing the point (shown as C in fig. 1) to a piece of plate glass which slides on top of the cast-iron box. The height of the point C above the plate is adjusted by fixing it to a micrometer screw B.

In order to record the position of the point C when contact with the film is made, the micrometer carries a recording point D, which points upwards and is placed exactly over C. The record is made on a sheet of paper fixed to the board E, which can swing about a horizontal axis at the same height as D. To mark any position of the screw it is merely necessary to prick a dot on the paper by bringing it down on the recording point. The process is repeated a large number of times, moving the point to different positions on the film but keeping the setting of the micrometer B constant. In this way a contour is pricked out on the paper. To prick out another contour the setting of B is altered. The photograph shows an actual record in which four contours traced in this way have been filled in with a pencil. The section shown is that of an aeroplane propeller blade.

To measure the inclination of the film to the plane of the plate the "auto-collimator" shown in figs. 2*a* and 2*b* was devised. Light from a small electric bulb A is reflected from the surface of the film through a V-neck B and a pin-hole eyepiece C placed close to the bulb.

Direct light from the bulb was kept away from the eye by a small screen. The inclinometer D, which measures the angle which the line of sight makes with the vertical, consists of a spirit level fixed to an arm which moves over a quadrant graduated in degrees. In using the auto-collimator the soap-film box is

adjusted till the plane of the hole across which the film is stretched is horizontal.

The volume contained between the film and the plane of the hole can be measured in a variety of ways. One of the most simple is to lay a flat glass plate wetted with soap solution over the test hole in such a way that all the air is expelled from it. The volume contained between the spherical film and plane of the circular hole is then increased by an amount equal to the volume required. This increase in volume can be determined in a variety of ways, one of the simplest being to make measurements with the auto-collimator of the inclination of the spherical film at a point on its edge.

Accuracy of the Method

Strictly speaking, the soap-film surface can only be taken to represent the torsion function if its inclination α is everywhere so small that $\sin\alpha = \tan\alpha$ to the required order of accuracy. This would mean, however, that the quantities measured would be so small as to render excessive experimental errors unavoidable. A compromise must therefore be effected. In point of fact, it has been found from experiments on sections for which the torsion function can be calculated, that the ratio of the stress at a point in any section to the stress at a point in a circular shaft, whose radius equals the value of $\frac{2A}{P}$ for the section, is given quite satisfactorily by the value of $\frac{\sin\alpha}{\sin\beta}$ where α and β are the respective inclinations of the corresponding films, even when α is as much as 35°. Similarly, the volume ratio of the films has been found to be a sufficiently good approximation to the corresponding torque ratio, for a like amount of displacement.

In contour mapping, the greatest accuracy is obtained, with the apparatus shown in fig. 1, when β is about 20°. That is to say, the displacement should be rather less than for the other two methods of experiment.

In all soap-film measurements the experimental error is naturally greater the smaller the value of $\frac{2A}{P}$. Reliable results cannot be obtained, in general, if $\frac{2A}{P}$ is less than about half an inch, so that a shape such as a rolled I beam section could not be treated satis-

factorily in an apparatus of convenient size. As a matter of fact, however, the shape of a symmetrical soap film is unaltered if it be divided by a septum or flat plate which passes through an axis of symmetry and is normal to the plane of the boundary. It is therefore only necessary to cut half the section in the test-plate and to place a normal septum of sheet metal at the line of division. An I beam, for instance, might be treated by dividing the web at a distance from the flange equal to two or three times the thickness of the web. It has been found advisable to carry the septum down through the hole so that it projects about $\frac{1}{8}$ in. below the under side of the plate, as otherwise solution collects in the corners and spoils the shape of the film.

TABLE I

SHOWING EXPERIMENTAL ERROR IN SOLVING STRESS EQUATIONS BY MEANS OF SOAP FILMS

Section.	Radius of Circle.	α.	β.	$\frac{\alpha}{\beta}$.	$\frac{\text{Sin}\alpha}{\text{Sin}\beta}$.	True Value.	Error, $\frac{\alpha}{\beta}$.	Error, $\frac{\sin\alpha}{\sin\beta}$.
	In.	Deg.	Deg.				Per Cent.	Per Cent.
1. Equilateral triangle: height, 3 in. ..	1·00	32·55	21·19	1·536	1·490	1·500	+2·4	−0·7
2. Square: side, 3 in. ..	1·5	29·11	21·34	1·364	1·337	1·350	+1·0	−1·0
3. Ellipse: semi-axes, 2 × 1 in.	1·296	30·71	24·32	1·263	1·240	1·234	+2·4	+0·5
4. Ellipse: 3 × 1 in. ..	1·410	31·10	24·00	1·296	1·270	1·276	+1·6	−0·5
5. Ellipse: 4 × 0·8 in.	1·196	35·35	26·58	1·331	1·293	1·286	+3·5	+0·5
6. Rectangle: 4 × 2 in.	1·333	31·70	22·36	1·418	1·380	1·395	+1·6	−1·1
7. Rectangle: 8 × 2 in.	1·60	34·83	27·23	1·279	1·247	1·245	+2·7	+0·2
*8. Infinitely long rectangle: 1 in. wide	1·00	36·42	36·19	1·006	1·005	1·000	+0·6	+0·5

The values set down in Table I indicate the degree of accuracy obtainable with the auto-collimator in the determination of the maximum stresses in sections for which the torsion function is known. They also give an idea of the sizes of holes which have been found most convenient in practice. The angles given are α, the maximum inclination at the edge of the test film, and β, the inclination at the

* On 4-in. length.

edge of the circular film of radius $\frac{2A}{P}$. They are usually the means of about five observations and are expressed in decimals of a degree.

The last two columns show the errors due to taking the ratio of angles and the ratio of sines respectively as giving the stress ratio.

The error is always positive for α/β, and its mean value is 1·98 per cent. In the case of $\frac{\sin\alpha}{\sin\beta}$ the average error is only 0·62 per cent. In only two instances does the error reach 1 per cent, and in both it is negative. The presence of sharp corners seems to introduce a negative error which is naturally greatest when the corners are nearest to the observation point. Otherwise, there is no evidence that the error depends to any great extent on the shape. Nos. 4, 5, 7, and 8 in the table are examples of the application of the method of normal septa described above in which the film is bounded by a plane perpendicular to the hole at a plane of symmetry.

Table II shows the results of volume determinations made on each of the sections 1 to 8 given in the previous table.

TABLE II

SHOWING EXPERIMENTAL ERROR IN DETERMINING TORQUES BY MEANS OF SOAP FILMS

No.	Section.	Maximum Inclination.	Observed Volume Ratio.	Calculated Torque Ratio.	Error.
		Deg.			Per cent.
1.	Equilateral triangle: height, 3 in. ..	32·06	1·953	1·985	−1·6
2.	Square: side, 3 in...	30·39	1·416	1·432	−1·1
3.	Ellipse: semi-axes 2 in. × 1 in. ..	30·50	1·143	1·133	+0·9
4.	Ellipse: 3 in. × 1 in.	31·01	2·147	2·147	—
5.	Ellipse: 4 in. × 0·8 in.	36·12	3·041	3·020	+0·7
6.	Rectangle: sides, 4 in. × 2 in. ..	31·33	1·456	1·475	−1·3
7.	Reactangle: 8 in. × 2 in.	35·28	1·749	1·744	+0·3
* 8.	Infinitely long rectangle	36·00	0·858	0·848	+1·2

* On 4-in. length.

The average error is 0·89 per cent. In four of the eight cases considered the error is greater than 1 per cent and in three of these it is negative. One may conclude that the probable error is somewhat greater than it is for the stress measurements, and that it tends to be negative. Its upper limit is probably not much in excess of 2 per cent. The remarks already made regarding the dependence of accuracy on the shape of the section apply equally to torque measurements.

When contour lines have been mapped, the torque may be found from them by integration. If the graphical work is carefully done, the value found in this way is rather more accurate than the one obtained by the volumetric method. Contours may also be used to find stresses by differentiation, that is, by measuring the distance apart of the neighbouring contour lines; but here the comparison is decidedly in favour of the direct process, owing to the difficulties inseparable from graphical differentiation. The contour map is, nevertheless, a very useful means of showing the general nature of the stress distribution throughout the section in a clear and compact manner. The highly stressed parts show many lines bunched together, while few traverse the regions of low stress, and the direction of the maximum stress is shown by that of the contours at every point of the section. Furthermore, the map solves the torsion problem, not only for the boundary, but also for every section having the same shape as a contour line.

Example of the Uses of the Method

The example which follows serves to illustrate the use of the soap-film apparatus in solving typical problems in engineering design.

It is well known that the stress at a sharp internal corner of a twisted bar is infinite or, rather, would be infinite if the elastic equations did not cease to hold when the stress becomes very high. If the internal corner is rounded off the stress is reduced; but so far no method has been devised by which the amount of reduction in strain due to a given amount of rounding can be estimated. This problem has been solved by the use of soap films.

An L-shaped hole was cut in a plate. Its arms were 5 in. long by 1 in. wide, and small pieces of sheet metal were fixed at each end, perpendicular to the shape of the hole, so as to form normal septa. The section was then practically equivalent to an angle with arms of

infinite length. The radius in the internal corner was enlarged step by step, observations of the maximum inclination of the film at the internal corner being taken on each occasion.

The inclination of the film at a point 3·5 in. from the corner was also observed, and was taken to represent the mean boundary stress in the arm, which is the same as the boundary stress at a point far from the corner. The ratio of the maximum stress at the internal corner to the mean stress in the arm was tabulated for each radius on the internal corner.

The results are given in Table III.

TABLE III

SHOWING THE EFFECT OF ROUNDING THE INTERNAL CORNER ON THE STRENGTH OF A TWISTED L-SHAPED ANGLE BEAM

Radius of Internal Corner.	Ratio: $\dfrac{\text{Maximum Stress}}{\text{Stress in Arm}}$.
Inches.	
0·10	1·890
0·20	1·540
0·30	1·480
0·40	1·445
0·50	1·430
0·60	1·420
0·70	1·415
0·80	1·416
1·00	1·422
1·50	1·500
2·00	1·660

It will be seen that the maximum stress in the internal corner does not begin to increase to any great extent till the radius of the corner becomes less than one-fifth of the thickness of the arms. A curious point which will be noticed in connection with the table is the minimum value of the ratio of the maximum stress to the stress in the arm, which occurs when the radius of the corner is about 0·7 of the thickness of the arm.

In fig. 3 is shown a diagram representing the appearance of these sections of angle-irons.

No. 1 is the angle-iron for which the radius of the corner is one-tenth of the thickness of the arm. This angle is distinctly weak at the corner.

In No. 2 the radius is one-fifth of the thickness. This angle-iron is nearly as strong as it can be. Very little increase in strength is effected by rounding off the corner more than this. No. 3 is the angle with minimum ratio of stress in corner to stress in arm.

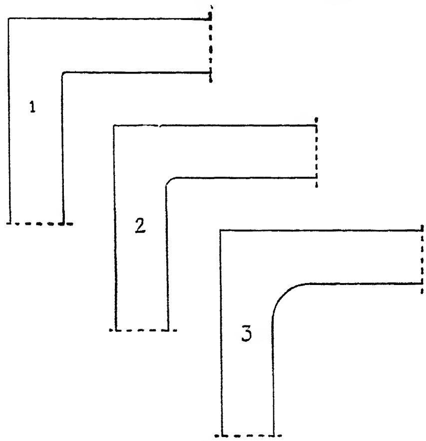

Fig. 3

A further experiment was made to determine the extent of the region of high stress in angle-iron No. 1. For this purpose contour lines were mapped, and from these the slope of the bubble was found at a number of points on the line of symmetry of the angle-iron. Hence the stresses at these points were deduced The results are given in Table IV.

TABLE IV

SHOWING THE RATE OF FALLING-OFF OF THE STRESS IN THE INTERNAL CORNER OF THE ANGLE-IRON

Distance from Boundary.	Ratio: $\frac{\text{Stress at Point}}{\text{Boundary Stress in Arm}}$.
Inches.	
0·00	1·89
0·05	1·36
0·10	1·12
0·20	0·77
0·30	0·49
0·40	0·24
0·50	0·00

It will be seen that the stress falls off so rapidly that its maximum value is to all intents and purposes a matter of no importance, if the material is capable of yielding. If the material is brittle and not ductile a crack would, of course, start at the point of maximum stress and penetrate the section.

Comparison of Soap-film Results with those obtained in Direct Torsion Experiments

As an example of the order of accuracy with which the soap-film method can predict the torsional stiffness of bars and girders of types used in engineering, a comparison has been made with the experimental results of Mr. E. G. Ritchie.* The torsional stiffness of any section can be represented by a quantity C such that torque $= CN\mathfrak{T}$, where N is the modulus of rigidity. C has dimensions $(\text{length})^4$. In Table V column 2 is given the value of C found by soap-film methods, while in column 3 is given the corresponding experimental results taken from Mr. Ritchie's paper.

TABLE V

Section.	C (Soap Film).	C (Direct Torsion Experiments).
Angle: 1·175 × 1·175 in.	0·01234 in.^{4}	0·01284 in.^{4}
Angle: 1·00 × 1·00 in.	0·0044 in.^{4}	0·00455 in.^{4}
Tee: 1·58 × 1·58 in.	0·01451 in.^{4}	0·01481 in.^{4}
I-beam: 5·01 × 8·02 in.	1·160 in.^{4}	1·140 in.^{4}
I-beam: 3·01 × 3·00 in.	0·1179 in.^{4}	0·1082 in.^{4}
I-beam: 1·75 × 4·78 in.	0·0702 in.^{4}	0·0635 in.^{4}
Channel: 0·97 × 2·00 in.	0·0175 in.^{4}	0·0139 in.^{4}

Torsion of Hollow Shafts

The method described above must be modified when it is desired to find the torsion function for a hollow shaft. In this case the function satisfies the equation (6) and the boundary conditions are $\Psi =$ constant on each boundary, but the constant is not necessarily the same for each boundary. In order to make use of the soap-film analogy it is therefore necessary to cut a hole in a flat sheet of metal to represent the outer boundary, and to cut a metal plate to represent the inner boundary. These are placed in the correct relative positions in the apparatus shown in fig. 1, and they are set so that they lie in parallel planes. The soap film is then stretched across the gap between them.

The planes containing the two boundaries must be parallel,

* *A Study of the Circular Arc Bow Girder*, by Gibson and Ritchie (Constable & Company, 1914).

but they may be at any given distance apart and yet satisfy the condition that $\Psi =$ constant round each boundary. On the other hand the contour lines of the film, and hence the value of Ψ, will vary greatly according to what particular distance apart is chosen. The solution of the torsion problem must be quite definite, so that must be possible to fix on the particular distance apart at which the planes of the boundaries must be set in order that the soap film stretched on them may represent the required torsion function. To do this it is necessary to consider again the function ϕ, which represents the displacement of a particle from its original position owing to the warping of plane cross sections of the twisted material. This function ϕ is evidently a single-valued function of x and y, i.e. it can have only one value at every point of the material. In general the values of Ψ found by means of the soap-film apparatus do not correspond with single-valued functions ϕ. On the other hand, there is one particular distance apart at which the planes of the boundaries can be placed so that the Ψ function does correspond with a single-valued function ϕ. To solve the torsion problem we must find this distance.

If ϕ is single-valued, $\int \frac{\partial \phi}{\partial s} ds = 0$ when the integral is taken round either boundary; and since $\frac{\partial \phi}{\partial s} = -\frac{\partial \psi}{\partial s}$, this condition reduces to $\int \frac{d\psi}{\partial n} ds = 0$. Substituting $\Psi = \psi - \frac{1}{2}(x^2 + y^2)$ and remembering that

$$\frac{1}{2}\frac{\partial}{\partial n}(x^2 + y^2) = \frac{\partial y}{\partial n}\frac{\partial}{\partial y}\left(\frac{x^2 + y^2}{2}\right) + \frac{\partial x}{\partial n}\frac{\partial}{\partial x}\left(\frac{x^2 + y^2}{2}\right) = y\frac{\partial x}{\partial s} - x\frac{\partial y}{\partial s},$$

it will be seen that

$$\int \frac{\partial \psi}{\partial n} ds = \int \frac{\partial \Psi}{\partial n} ds - 2A, \qquad \ldots\ldots\ldots\ldots(14)$$

where A represents the area of the boundary. The condition that ϕ shall be single-valued is therefore

$$\int \frac{\partial \Psi}{\partial n} ds = 2A. \qquad \ldots\ldots\ldots\ldots\ldots(15)$$

Referring again to the soap-film analogy, and putting $\Psi = 4\gamma z/p$, it will be seen that (15) is equivalent to

$$2S\int \sin a\, ds = Ap. \qquad \ldots\ldots\ldots\ldots\ldots(16)$$

Equation (16) applies to either boundary; it may be compared with equation (12), which there applies only to a solid shaft. Taking the case of the inner boundary, it will be noticed that Ap is the total pressure exerted by the air on the flat plate which constitutes the inner boundary. $2\gamma\int \sin a ds$ * on the other hand is the vertical component of the force exerted by the tension of the film on the inner boundary. Hence the condition that ϕ shall be single-valued gives rise to the following possible method of determining the position of the inner boundary. The plate representing it might be attached to one arm of a balance. The film would then be stretched across the space between the boundaries, and if the outer boundary was at a lower level than the inner one the tension in the film would drag the balance down. The pressure of the air under the film would then be raised till the balance was again in equilibrium. The film so produced would satisfy condition (16).

As a matter of fact this method is inconvenient, and another method based on the same theoretical principles is used in practice, but for this and further developments of the method to such questions as the flexure of solid and hollow bars the reader is referred to Mr. Griffith's and Mr. Taylor's papers published in 1916, 1917, and 1918 in the Reports of the Advisory Committee for Aeronautics.

Example of the Application of the Soap-film Method to Hollow Shafts

As an example of the type of research to which the soap-film method can conveniently be applied, a brief description will be given of some work undertaken to determine how to cut a keyway in the hollow propeller shaft of an aeroplane engine, so that its strength may be reduced as little as possible. These shafts used to be cut with sharp re-entrant angles at the bottom of the keyway, and they frequently failed owing to cracks due to torsion which started at the re-entrant corners. It was proposed to mitigate this evil by putting radii or fillets at these corners, and it was required to know what amount of rounding would make the shafts safe.

The shafts investigated were 10 in. external and 5·8 in. internal diameter. This was not the size of the actual shafts used in aero-

* The factor 2 comes in owing to the fact that γ is the surface tension of one surface and the film has two surfaces.

planes, but it was found to be the size which gave most accurate results with the soap-film apparatus.

Some of the results of the experiments are shown graphically in the curve in fig. 4.* In this curve the ordinates represent the maximum

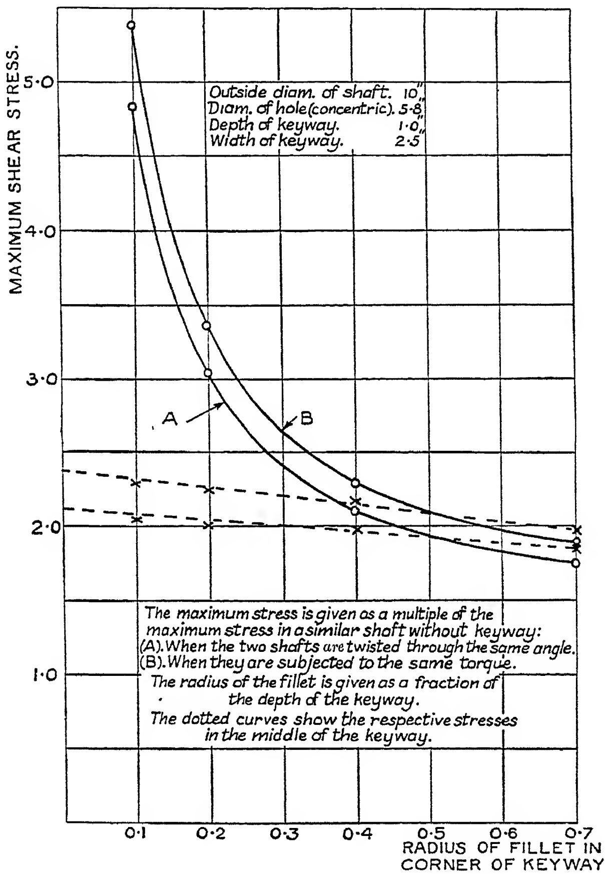

Fig. 4.—Torsional Strength of Hollow Shaft with Keyway

shear stress, on an arbitrary scale, while the abscissæ represent the radius of the fillet, in which the internal corners of the keyway were rounded off. It will be seen that the shaft begins to weaken rapidly when the radius is less than about 0·3 in..

* This diagram and also that shown in fig. 5 are taken from Messrs. Griffith and Taylor's report to the Advisory Committee for Aeronautics, 1918.

The lines of shearing stress, i.e. the contour lines of the soap film, are shown in fig. 5 for the case when the radius of the fillet is 0.2 in.

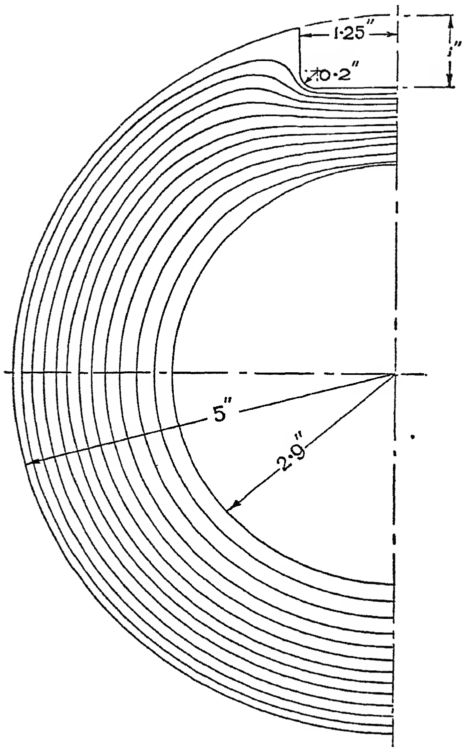

Fig. 5.—Lines of Shearing Stress in the Torsion of a Hollow Shaft with Keyway

It will be seen that the lines of shearing stress are crowded together near the rounded corner of the keyway.

CHAPTER VIII

Wind Structure

During the present century great advances have been made in the field of aviation, and problems, some of them entirely new, others under a new guise, have presented themselves. Among the latter may be included the problem of *wind structure*. Slight changes of the wind, both in direction and in magnitude, are of little account for some problems where only the average effect of the wind is of any moment. On the other hand, for the aviator these small changes are often of far greater moment than the general drift, especially when his machine is either leaving or approaching the ground. Now it is just at this point that the irregularities are often greatest.

Before proceeding to deal with the cause of these various irregularities, let us consider what are the governing factors in the movement of a mass of air over the surface of the globe.

Apart from the difficulties of dynamics, the general problem is one of much complexity. In the first place, the surface of the earth is not at all uniform. It consists of land and water surfaces, and a land surface and a water surface behave quiet differently towards solar radiation, so that air over one area becomes more warmed up than that over another. Further, the land areas are divided into deserts and regions rich in vegetation, flat plains, and mountain ranges. Again, water vapour, to whose presence in the atmosphere nearly all meteorological phenomena are due, while being added at one place, is not subtracted simultaneously at another, so that the amount present in the atmosphere varies very irregularly. These and other factors tend to render an exact mathematical solution of the problem practically impossible.

An approximate determination, however, of the effect of the earth's rotation on the horizontal distribution of pressure when the air moves over the surface of the globe in a simple specified manner, can be found.

To obtain this approximate solution of the problem, we shall

assume that the air is moving horizontally* with constant linear velocity v, i.e. that a steady state has been reached. The forces acting on a particle of air, in consequence of its motion, under these conditions at a place P in latitude ϕ arise from two causes, (1) the rotation of the earth, and (2) the curvature of the path in which the particle is moving at the instant, relative to the earth. The problem before us therefore is (1) to find the magnitude of the accelerations arising from these causes and (2) to show how the forces required for these accelerations in the steady state are provided by the pressure gradient.

Consider first the effect of the rotation of the earth on great-circle motion. We shall suppose the particle is constrained, by properly adjusted pressure gradients, to move in a great circle through P with uniform velocity. The particle is therefore supposed to move in a path which is rotating in space about an axis passing through its centre.

The rotation of the earth takes places about its axis NS, fig. 1. The great circle Q'PQ is the specified path of the particle. The earth's rotation may be resolved by the parallelogram of rotations into two rotations about any two directions in a plane containing NS. Let these two directions be the two perpendicular lines OP and OW′, where O is the centre of the earth and P the point in latitude ϕ referred to above. If the angular velocity of the earth about SN be ω, then the component angular velocities are $\omega \cos\phi$ about OW′ and $\omega \sin\phi$ about OP. As the two axes are mutually perpendicular, it follows that any particle in the neighbourhood of P is in the same relation to OW′ as a particle on the equator is to ON. But a particle on the equator moving with uniform horizontal velocity has an acceleration directed *only perpendicular* to the axis ON, and therefore its horizontal velocity is not affected by the rotation about ON. Similarly the horizontal velocity of a particle near P is affected only by the component $\omega \sin\phi$ about OP, and not by the perpendicular component $\omega \cos\phi$ about OW′. We need consider therefore only the effect of the component $\omega \sin\phi$.

When the particle crosses the point P, it will travel a distance PA $= vdt$ (see fig. 1*a*) in time dt, as the velocity is v. In the same interval of time, the line along which the particle started will have moved into the position PA′, so that the element of arc $ds =$ AA $=$ PA$\omega \sin\phi dt$.

* I.e. in a plane perpendicular to the direction of the force compounded of the force of gravity and the centrifugal force.

Also ds or AA′, which is described in a direction perpendicular to PA, may, by the ordinary formula, be expressed in the form

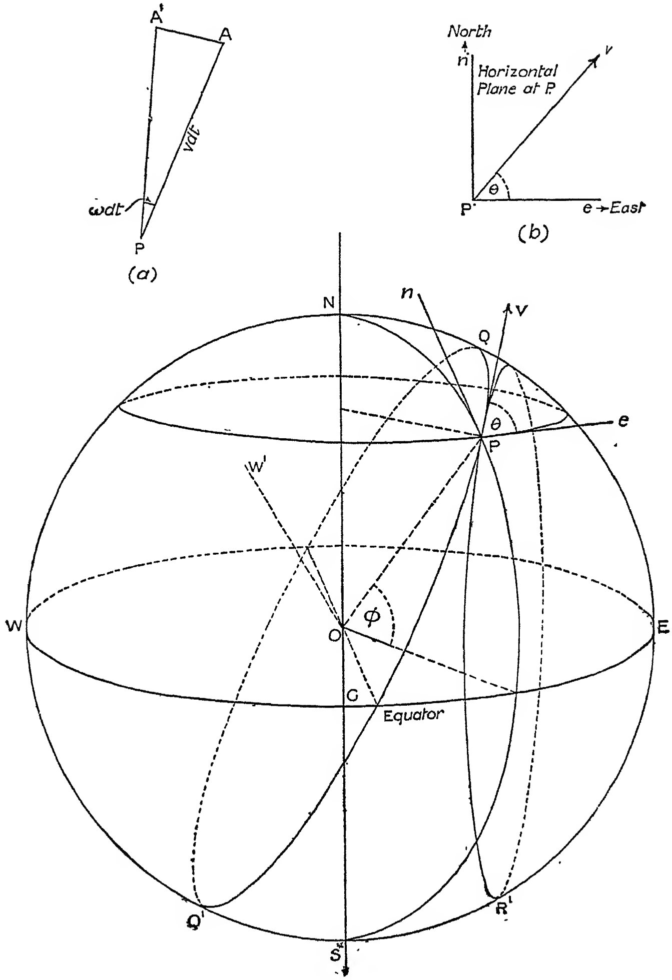

Fig. 1

$\frac{1}{2}f(dt)^2$, where f is an acceleration in the direction perpendicular to the direction of motion. Hence

$$\tfrac{1}{2}f(dt)^2 = \mathrm{PA}\omega\sin\phi dt = v\omega\sin\phi(dt)^2,$$

since $\mathrm{PA} = vdt;$

i.e. $f = 2v\omega\sin\phi.$

Hence the transverse force (F) necessary to keep a mass (m) of air moving along a great circle, in spite of the rotation of the earth, is given by

$$F = mf = 2mv\omega \sin\phi, \quad \ldots\ldots\ldots\ldots\ldots (1)$$

acting, in the northern hemisphere, towards the left, in the southern towards the right, when looking along the direction of the wind.

This expression, which is very nearly correct, shows that the deflective force due to the rotation of the earth* on a mass of moving air is (1) directly proportional to the mass, to the horizontal velocity, to the earth's angular velocity, and to the sine of the latitude of the place; (2) independent of the direction of the great circle, i.e. of θ (fig. 1); (3) always perpendicular to the instantaneous direction of motion of the air and therefore without influence on the velocity with reference to the surface; (4) opposite to the direction of the earth's rotation.

When the air moves, as specified, in a great circle, the acceleration $2v\omega \sin\phi$ is the only transverse acceleration in the horizontal plane, for the acceleration arising from the curvature of the path relative to the earth (which exists even if ω were zero) is radial and therefore has no appreciable component in the horizontal plane.

Now suppose that the path is not a great circle but a small one, R′P (fig. 1). In addition to the term $2v\omega \sin\phi$ there will now be a term arising from the curvature of the path. This term is independent of ω. Let fig. 2 be a section of the sphere through a diameter of the small circle, PR′ being the diameter. The path of the air at P is now curved, and if r is the radius of curvature of the path at P, in the horizontal plane, $\frac{v^2}{r}$ is the acceleration in the horizontal plane arising from the curvature of the path. But this acceleration is also the horizontal component of $\frac{v^2}{r'}$ where r' is PM, i.e. the radius of curvature of the small circle. If α is the angular radius of the small circle it is also the inclination of the horizontal plane to the plane of the small circle (see fig. 2), hence

$$\frac{v^2}{r} = \frac{v^2}{r'} \cos\alpha;$$

$$\therefore r' = r \cos\alpha,$$

* I.e. the force F, *reversed.*

i.e. N (fig. 2) is the centre of curvature of the path in the horizontal plane.

It is also clear from fig. 2 that R $\sin\alpha = r'$, hence

$$\frac{v^2}{r} = \frac{v^2}{r'}\cos\alpha = \frac{v^2}{R\sin\alpha}\cos\alpha = \frac{v^2}{R}\cot\alpha.$$

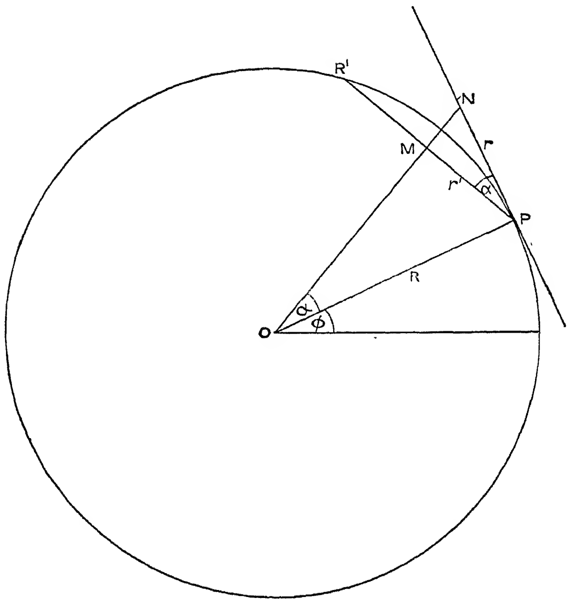

Fig. 2.—Relation between Radii of Curvature of the Path on the Earth and the Path in the Horizontal Plane. PM $= r'$. PN $= r'/\cos\alpha = r$. PO $=$ R

If the air is moving freely in space, i.e. if the barometric pressure is uniform, the resultant horizontal acceleration is zero, i.e.

$$2v\omega\sin\phi + \frac{v^2}{R}\cot\alpha = 0.$$

Hence " free " motion is only possible when one of these accelerations has the opposite direction to the other, and

$$2v\omega\sin\phi = \frac{v^2}{R}\cot\alpha$$

numerically, i.e. when the acceleration due to path curvature balances that due to the earth's rotation.

When the barometric pressure is not uniform, we proceed thus:

If the particle or element of air at P occupies the volume of a small cylinder, of length δn in the direction of the *outward* drawn normal at P to the path of the air in the horizontal plane, and of unit cross-sectional area, the force on the air in the *inward* direction due to variation of pressure is $\left(\frac{\partial p}{\partial n}\delta n\right)$. The mass of air is $\rho\delta n$ where ρ is the density at P, hence

$$\frac{\partial p}{\partial n}\delta n = \rho\delta n\left[2v\omega\sin\phi + \frac{v^2\cot\alpha}{R}\right].$$

$$\therefore \frac{\partial p}{\partial n} = 2\rho v\omega\sin\phi + \frac{\rho v^2\cot\alpha}{R}. \qquad \ldots\ldots\ldots\ldots(2)$$

The formula is true for positive and negative values of v, remembering that $\frac{\partial p}{\partial n}$ is the gradient of pressure in the outward direction of the normal, and that the rotational term in the acceleration is towards the left hand when looking along the direction of the wind in the northern hemisphere. The two cases of *cyclonic* and *anticyclonic* wind (i.e. $+v$ and $-v$) are shown in fig. 3. The forces indicated in this figure are those required to keep the air in its assumed path, relative to the earth. These forces are provided by the pressure gradient. If we take the *numerical* value of the pressure gradient and the wind speed, then

$$\frac{\partial p}{\partial n} = 2\rho v\omega\sin\phi + \frac{\rho v^2}{R}\cot\alpha \qquad \ldots\ldots\ldots\ldots(3)$$

for the *cyclone*, where $\frac{\partial p}{\partial n}$ is the rate of rise of pressure outwards. For the anticyclone,

$$\frac{\partial p}{\partial n} = 2\rho v\omega\sin\phi - \frac{\rho v^2}{R}\cot\alpha, \qquad \ldots\ldots\ldots\ldots(3a)$$

where $\frac{\partial p}{\partial n}$ is the rate of rise of pressure *inwards*. Both cases are included in (2) without ambiguity.

These expressions give a value of the wind velocity called the *gradient wind* velocity. The direction of this gradient wind according to the previous reasoning is *along* the isobars, and is such that to one moving with it in the northern hemisphere, the lower pressure is on the left hand. It must be distinctly understood that in the above expressions for the gradient wind a steady state has been reached; and further, it is assumed in arriving at these expressions that there

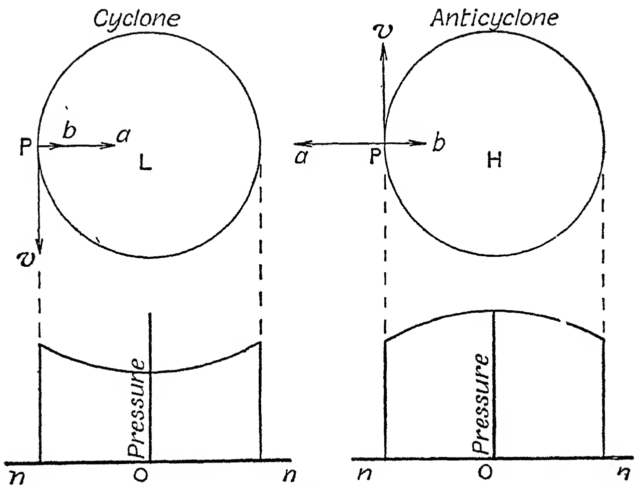

Fig. 3

$$Pa = 2\rho v\omega \sin\phi$$
$$= \text{term arising from rotation of the earth.}$$
$$Pb = \frac{\rho v^2 \cot\alpha}{R}$$
$$= \text{term arising from the angular radius of the path.}$$

is no friction between the air and the surface of the earth over which it is passing.

Under actual conditions the relation cannot be satisfied exactly as there is always a certain amount of momentum absorbed from the stream of air by the friction at the surface. This absorption of energy is manifested by the production of waves and similar effects on water surfaces, on forests, and on deserts. Yet under the most unfavourable conditions this relation between wind and pressure can be recognized, and therefore it must be an important principle in the structure of the atmosphere. Also when we ascend into the atmosphere beyond the limits where the influence of surface friction is likely to be felt, we find very little difference in the velocity of the

wind for hours on end. According to Shaw,* "pressure distribution seems to adjust itself to the motion of the air rather than to speed it or stop it. So it will be more profitable to consider the *strophic* balance between the flow of air and the distribution of pressure as an axiom or principle of atmospheric motion." This axiom he has enunciated as follows:† "In the upper layers of the atmosphere the steady horizontal motion of the air at any level is along the horizontal section of the isobaric surface at that level, and the velocity is inversely proportional to the separation of the isobaric lines in the level of the section."

Throughout this short study of wind structure we shall follow Shaw therefore, and regard the wind as balancing the pressure gradient. It may be argued that this assumption strikes at the root of the processes and changes in pressure distribution one may desire to study. The results of investigation appear to indicate, however, that in the free atmosphere, at all events, the balance is sufficiently good under ordinary conditions for us to take the risk and accept the assumption. Under special circumstances and in special localities there may occur singular points where the facts are not in agreement with the assumption, but the amount of light which can be thrown upon many hitherto hidden atmospheric processes, appears to justify our acceptance of it.

In the expression for the calculation of the gradient wind the right-hand side consists of two terms. The first term, $2\rho v w \sin\phi$, is due as we have seen to the rotation of the earth, and in consequence has been called the *geostrophic* component of the pressure gradient. The other part, $\frac{v^2\rho}{R}\cot\alpha$, arises from the circulation in the small circle of angular radius α, and so has been termed the *cyclostrophic* component. With decrease in ϕ, i.e. the nearer we approach the equator, α remaining constant, the first component therefore becomes less and less important, the balance being maintained by the second term alone practically. On the other hand, with increase in α, i.e. the nearer we approach to the condition of the air moving in a great circle, ϕ remaining constant, the second term becomes less and less important, until finally with the air moving on a great circle the gradient and the geostrophic wind are one and the same. Consequently in the pressure distributions in mean latitudes where the radius of curvature of the path is

* *Manual of Meteorology*, Part IV, p. 90.
† *Proc. Roy. Soc. Edin.*, **34**, p. 78 (1913).

generally very large, the geostrophic wind is commonly taken as the gradient wind.

Having obtained expressions indicating the connection between the pressure gradient and the theoretical velocity of the wind, we shall now consider some of the reasons for the variations of the wind velocity from this theoretical value.

In the equation for the gradient wind and in the statements made regarding the effects of friction on the wind, there is nothing to indicate that the flow of air is not steady. But it is a perfectly well

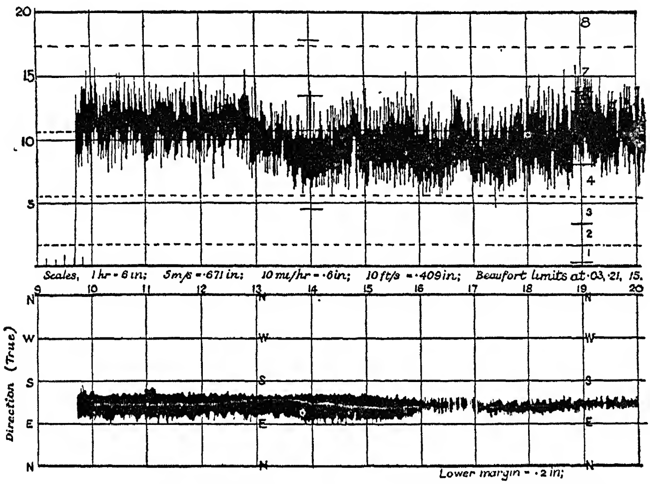

Fig 4.—Anemometer Record at Aberdeen Observatory, 26th September, 1922

known fact that the air, at all events near the surface, does not flow with a constant velocity even for a very short interval of time. This unsteadiness in the wind velocity is exhibited very well by the records of self-recording anemometers. Fig. 4, which is part of the record for 26th September, 1922, at Aberdeen, exhibits this moment-to-moment variation. Not only does the velocity vary but the direction also shows a similar variation, as indicated by the lower trace in the figure. As a rule the greater the variation in velocity, the greater also the variation in direction.

The variations both in velocity and direction are very largely dependent upon the nature of the surface over which the air is passing, i.e. the nature of the records is greatly affected by the exposure of the anemometer. A comparison of figs. 4 and 4*a* reveals this very plainly. The first, as stated above, is a record from the

anemometer at King's College Observatory, Aberdeen. The head of the instrument is 40 ft. above the ground, the instrument itself being housed in a small hut * in the middle of cultivated fields and placed about $\frac{1}{2}$ mile from the sea. The second is a record from an anemometer situated about 5 miles inland from the first, at Parkhill Dyce, and belonging to Dr. J. E. Crombie. The exposure in this case is over a plantation of trees, and though the head of the instrument is 75 ft. above the ground, it is only 15 ft. above the level of

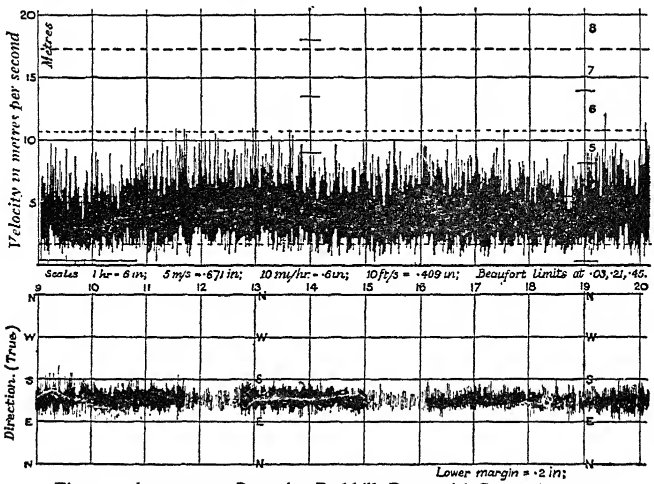

Fig. 4a.—Anemometer Record at Parkhill, Dyce, 26th September, 1922

the tree-tops. The two records refer to the same day, and the anemometers are situated comparatively close the one to the other, yet the "gustiness" as indicated on the second is much greater than that on the first. At the same time the average velocity of the wind in the second case is considerably reduced by the general effects of the nature of the exposure.

Other factors which affect this variation of the wind are to be found in the undisturbed velocity of the wind in the upper air and in the temperature of the surface of the ground, i.e. in the time of day and in the season of the year. A great deal of light has been thrown on these variations of the wind near the surface by G. I. Taylor through his investigations of eddy motion in the atmosphere.†

* The position of this hut is about a quarter of a mile directly eastwards from the position of the anemometer shown in fig. 6. Fig. 6 is compiled from records taken in the old position.

† "Phenomena connected with Turbulence in the Lower Atmosphere", *Proc. Roy. Soc.* A, 94, p. 137 (1918).

From the aviator's point of view these variations are often of prime importance. Beginning therefore at the surface, we shall endeavour to ascertain how the actual wind is related to the geostrophic or gradient wind for various exposures, and afterwards determine how these relations alter as we ascend higher into the atmosphere.

When the hourly mean values of the surface wind velocities are examined for any land station, it is found that there is a diurnal

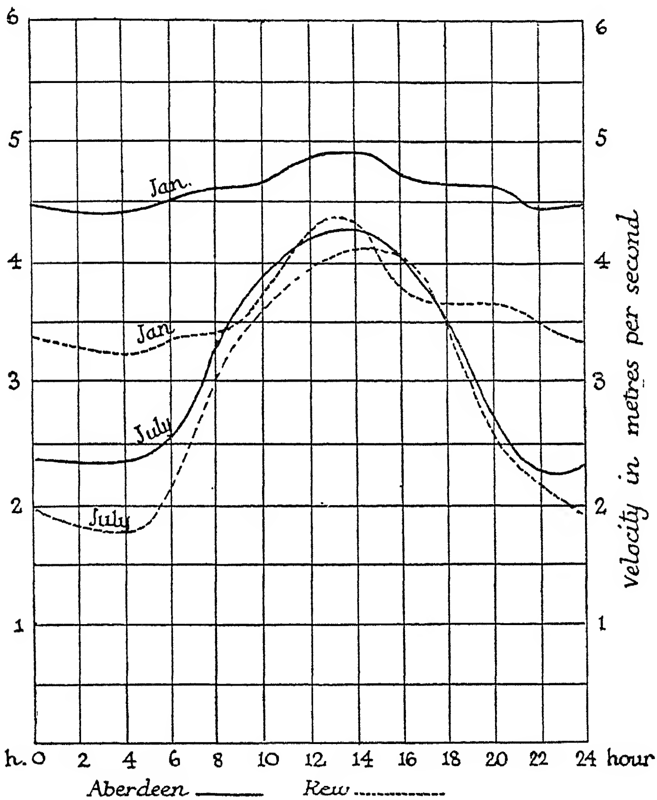

Fig. 5.—Diurnal Variation in Wind Velocity for January and July, for the period 1881–1910

and also a seasonal variation in the velocities. Fig. 5 represents this diurnal variation for the two stations, Aberdeen and Kew, for the months of January and July. On the other hand, no corresponding diurnal variation of the barometric gradient is to be found for these stations. The diurnal variation of the wind is evidently

dependent upon the diurnal and seasonal variations of temperature, and therefore the relation of the surface wind to the geostrophic wind is also dependent on these quantities. The curves in fig. 5 show a maximum corresponding closely with the time of maximum

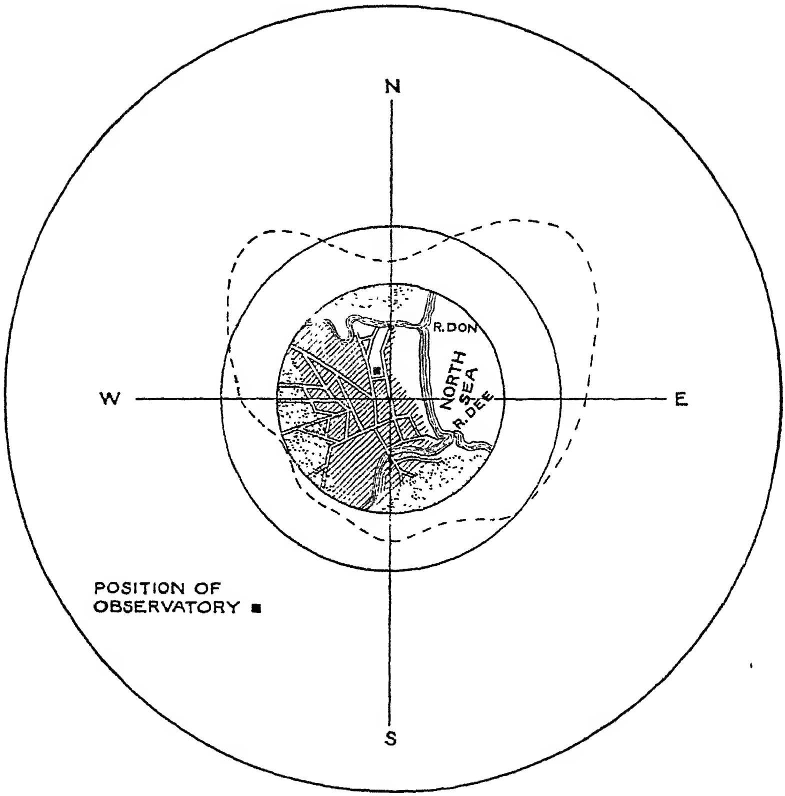

Fig. 6.—Relation between Geostrophic and Observed Surface Winds of Force 4 at Aberdeen

Central circle shows position of the city of Aberdeen with reference to the two river valleys and the sea. Stippled area shows high ground. Hatching area shows town buildings.

The outer circle represents the gradient wind. The inner circle represents 43 per cent of the gradient wind. The dotted line represents the observed wind.

temperature. Therefore if W represent the surface wind and G the geostrophic wind, the ratio W/G increases with increase of temperature, and vice versa. If then the surface layers be warmed or cooled from any cause whatsoever, we always find this effect on the ratio W/G.

The exposure of a station also has its effect on the ratio W/G.

TABLE I

THE RELATION OF SURFACE WIND TO GEOSTROPHIC WIND. W/G EXPRESSED AS A PERCENTAGE

Station.	Nature of Site.	Height of Ground.	N.	N.N.E.	N.E.	E.N.E.	E.	E.S.E.	S.E.	S.S.E.	S.	S.S.W.	S.W.	W.S.W.	W.	W.N.W.	N.W.	N.N.W.	Mean.
Stornoway	Sloping shore of bay	51 ft.	*48*	57	**63**	58	51	52	*48*	49	60	57	52	51	50	*48*	49	50	53
Malin Head ..	Headland knoll ..	208 „	72	**73**	60	39	65	66	51	*44*	46	48	48	56	57	67	70	68	58
Cahirciveen ..	Mouth of glen ..	30 „	58	57	55	47	55	42	39	38	*37*	40	41	54	54	**60**	59	57	50
Holyhead	Flat island	15 „	85	68	59	57	53	37	*34*	49	51	61	51	48	60	69	79	**87**	59
St. Mary's, Scilly ..	Hilly island	118 „	**74**	63	59	59	67	70	59	47	51	*45*	49	49	54	63	65	61	58
Portland Bill ..	Low point of headland	19 „	65	68	73	**80**	68	56	53	53	*48*	51	56	59	64	67	71	73	63
Aberdeen	College roof	46 „	35	45	**61**	57	51	45	43	35	36	32	32	*31*	39	46	54	43	43
Spurn Head ..	Spit of sand	26 „	80	**85**	64	72	65	72	64	54	*49*	*49*	*49*	53	59	76	76	73	65
Paisley	Inland station ..	—	37	33	53	**53**	47	33	*28*	35	33	35	36	36	41	36	42	47	38
Woburn	Inland station ..	—	49	49	54	57	40	45	**61**	42	*31*	43	43	44	49	50	49	46	46

Note.—The observations in every case refer to winds of force 4, and extend over 8 years. The highest and lowest percentages are indicated respectively by black type and italic figures.

Fig. 6 shows this effect on winds of force 4 at Aberdeen. The geostrophic wind is represented by the outer circle, while the dotted irregular curve gives the percentage which the surface wind is of the geostrophic wind. An idea of the exposure of the station is afforded by the circular portion of the ordnance map of the district placed at the centre of the figure. On the west side there is land, on the east, sea. To the south-west of the station lies the city, and we find that in this direction the surface wind has the lowest percentage, while in the north-easterly directions the percentages are largest. Towards the north-west lies the valley of the Don, and a fairly open exposure, the effects of which are also well brought out in the figure.

The effect of different exposures on winds of the same geostrophic magnitude will be understood readily from an examination of Table I.

It is evident, therefore, that no general rule can be given with regard to the value of the ratio W/G. It may have a wide range from approximately unity downwards, depending on the time of day, the season of the year, and the exposure of the station. In the same way the deviation α of the surface wind from the direction of the geostrophic wind is found to vary over a wide range.

An example of this is afforded by Table II, wherein are set out the values for Pyrton Hill and Southport, as given by J. S. Dines in the *Fourth Report on Wind Structure* to the Advisory Committee on Aeronautics.

It is necessary, therefore, in giving an estimate from the barometric gradient of the probable surface wind, as regards either direction or velocity, that due attention be paid to the details mentioned above.

Occasionally the surface wind is found to be in excess of the gradient. This probably arises from a combination of a katabatic* effect with the effect of the pressure distribution, the katabatic effect more than compensating for the loss of momentum in the normal wind due to friction at the earth's surface.

Data for the purpose of examining the ratio W/G over the sea are very limited. The following table, as given in the Meteorological Office report for moderate or strong winds over the North Sea, will serve to show the deviation of the surface wind from the gradient wind, both in velocity and direction.

* I.e. Katabatic or Gravity Wind: when the surface air over a slope cools at night or from any other cause it tends to flow down the slope; this is especially pronounced on clear nights. In ravines, if snow-covered and devoid of forests, this wind often reaches gale force. Such a wind is known as a katabatic wind.

TABLE II

	Geostrophic Wind Direction.	N.	N.N.E.	N.E.	E.N.E.	E.	E.S.E.	S.E.	S.S.E.	S.	S.S.W.	S.W.	W.S.W.	W.	W.N.W.	N.W.	N.N.W.
		Deg.	Deg.	Deg.	Deg.	Deg.	Deg.	Deg.	Deg.	Deg.	Deg.	Deg.	Deg.	Deg.	Deg.	Deg.	Deg.
Pyrton Hill	Deviation α	29	24	11	13	22	27	31	37	40	35	37	43	37	42	45	34
Southport	,,	36	31	30	29	30	36	46	56	61	62	59	49	45	46	45	41

Mean surface-wind velocity (1) Pyrton Hill = 42·6 per cent of geostrophic wind.
(2) Southport = 51·6 ,, ,, ,,

Mean deviation of surface wind from geostrophic wind (1) Pyrton Hill = 32°.
(2) Southport = 44°.

TABLE III

Relation of Surface Wind to Geostrophic Wind (between 8·5 m/s and 18 m/s) over the North Sea

1. Percentage Frequency of Points in the Veer of the Geostrophic Wind from the Surface Wind

Deviation in points ..	− 4	− 3	− 2	− 1	0	+ 1	+ 2	+ 3	+ 4	+ 5	+ 6	+ 7
N.W. Quadrant ..	0	1	0	2	21	32	**33**	14	1	0	0	0
S.W. ,, ..	0	2	4	0	23	**30**	19	14	7	2	2	0
S.E. ,, ..	2	2	2	2	16	**29**	16	19	9	1	1	0
N.E. ,, ..	0	2	2	0	2	**31**	27	22	3	0	3	3

2. Percentage Frequency of the Ratios of Surface Wind to Geostrophic Wind within assigned limits

Limits of ratio ..	0·24–0·36	0·36–0·48	0·48–0·60	0·60–0·72	0·72–0·84	0·84–0·96	0·96–01·08
N.W. ,, ..	1	5	16	**38**	29	10	0
S.W. ,, ..	8	16	**34**	29	12	2	1
S.E. ,, ..	13	19	**26**	18	18	2	4
N.E. ,, ..	9	20	9	**29**	20	6	9

Note.—The maximum values of both direction and velocity for each quadrant have been indicated by black figures.

Here again we see that no *definite* rule can be given for estimating the surface wind from the geostrophic wind. It will be observed, however, that each quadrant exhibits certain dominant features and must be considered therefore by itself. In this way a considerable amount of guidance is obtained by the forecaster in estimating the wind over the sea from a given pressure distribution.

We now pass to consider the actual wind in relation to the geostrophic wind in the first half-kilometre above the surface. W. H. Dines, in his investigation on the relations between pressure and temperature in the upper atmosphere, has found a high correlation between the variations of these elements from their normal values for heights from 2 Km. upwards. Below the 2-Km. level the correlation coefficients gradually diminish until at the surface practically no connection at all is found. Above the 2-Km. level we may regard the air as in an " undisturbed " condition, i.e. free from the effect of the friction at the earth's surface. In this undisturbed region the velocity and direction of the wind at any given height are governed by the pressure and the temperature gradients ruling at that height, while in the lower layers we find considerable deviation from this law, evidently due to the effects of the surface of the earth on the air in contact with it.

Several empirical formulæ have been given whereby the velocity of the wind at any height in the lower layers of the atmosphere may be calculated from that at a definite height, say 10 m., above the earth's surface. From observations, up to 32 m., over meadowland at Nauen, Hellmann* confirmed an empirical formula $v = kh^{\frac{1}{5}}$, which agrees very nearly with a formula $v = kh^{\frac{1}{4}}$ suggested by Archibald† from kite observations in 1888. The results of observations up to 500 m., carried out in 1912 with two theodolites, are given by J. S. Dines in the *Fourth Report on Wind Structure* already referred to. Here he has represented his conclusions by a series of curves, and in doing so has grouped the winds into three sets: (1) very light, where the velocity at 500 m. is less than 4 m. per second; (2) light, with velocity between 4 m. per second and 10 m. per second; and (3) strong, with velocity greater than 10 m. per second. The curve for very light winds (see fig. 7) shows that in this class the surface wind approaches the geostrophic value, which is also marked for each group at the top of the diagram, much more closely than in any of the other groups. Curves of this type enable

* *Meteor. Zeitschrift*, 1915.
† *Nature*, 27, p. 243.

one to judge of the average behaviour of the wind in the lowest half-kilometre according to the pressure gradient at the surface. When, however, curves are drawn for different hours of the day, 7 hr., 13 hr., 18 hr., these show differences among themselves even for the same surface gradient. A whole series of curves for various hours of the day and different seasons of the year would be necessary, therefore, before a complete solution of the problem could be obtained.

As these formulæ and curves just referred to are applicable under certain conditions only, and as the constants used differ for

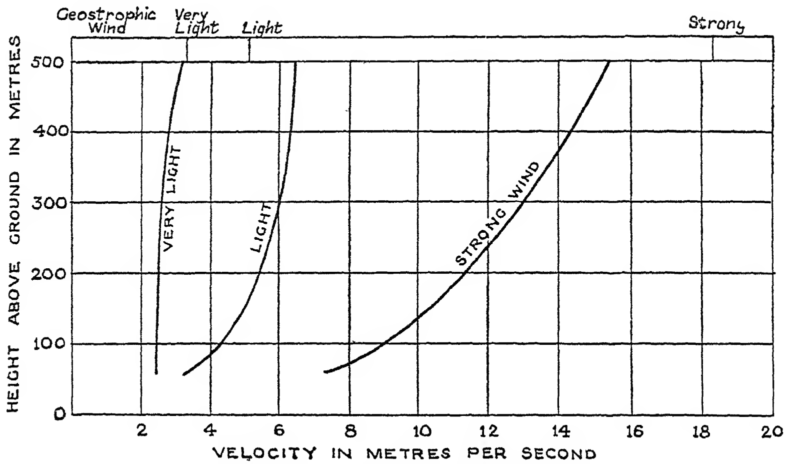

Fig. 7.—Change of Wind Velocity with Height within 500 metres above the surface

different times of the day and different seasons of the year, though they supply a rough working rule, yet a more exact solution of the problem is desirable. This has been supplied by the investigations of G. I. Taylor.* In his solution he regards the wind in the undisturbed layer as equivalent to the geostrophic wind at the surface, while the region between the surface and the undisturbed layer is considered as a slab through which the momentum of the undisturbed layer is propagated, as heat is conducted through a slab of material the two faces of which are kept at different temperatures. The momentum is propagated, according to the theory, by eddy motion, the surface of the earth acting as a boundary at which the momentum is absorbed. The equation representing the propagation is given as

$$\rho \partial u/\partial t = \frac{\partial}{\partial z}\left(\kappa\rho\frac{\partial u}{\partial z}\right), \quad \ldots\ldots\ldots\ldots\ldots (4)$$

* "Eddy Motion in the Atmosphere", *Phil. Trans.* A, **215**, p. 1 (1915).

where ρ = the density and κ = the "eddy conductivity" of the air. For small heights up to 1 Km or thereby, ρ and κ are approximately constant. Therefore the equation, representing the distribution of velocities with height and time within this region, may be written as

$$\rho \partial u/\partial t = \kappa \rho \partial^2 u/\partial z^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

The value of κ is, according to Taylor,* roughly $\frac{1}{2}wd$ where w is the mean vertical component of the velocity due to the turbulence, and d represents approximately the diameter of a circular eddy.

The value of κ differs, however, according to (1) the nature of the surface over which the air current is passing, (2) the season of the year, and (3) the time of day. As both heat and momentum are conducted by the eddies, the value of κ will be the same for both. Values of κ have accordingly been determined by Taylor as follows:

(1) Over the sea (determined from temperature observations over the Banks of Newfoundland)	3×10^3 C.G.S. units.
(2) Over grassy land (determined from velocity observations by pilot balloons over Salisbury Plain)	5×10^4 C.G.S. units.
(3) Over land obstructed by buildings (determined from the daily range of temperature observations at different levels on the Eiffel Tower)	10×10^4 C.G.S. units.

The effect of the season of the year on the value of κ is seen by comparing the values obtained from the Eiffel Tower observations for January and June.

Whole range, 18 to 302 m.:

(1) January	$4{\cdot}3 \times 10^4$	C.G.S. units.
(2) June	$18{\cdot}3 \times 10^4$	„
(3) Whole year	10×10^4	„

That κ is also to a certain extent dependent upon the height may be understood by comparing the values for the first stage, 18 to 123 m., with those for the last, 197 to 302 m.

Mean value for the whole year:

(1) Lowest stage	15×10^4	C.G.S. units.
(2) Highest stage	11×10^4	„

The reason for this variation is to be found in the method used in calculating κ. The nearer the ground the greater the daily

* *Proc. Roy. Soc.* A, **94**, p. 137 (1917).

variation of temperature, and therefore the error arising from the method used in the calculation is proportionately greater for the lower stages than for those higher up. The value of κ for the lowest stage is therefore not likely to be so accurate as that for the highest.

From wind measurements Åkerblom* has deduced the value of κ for the whole range of the Eiffel Tower. The value found, $7 \cdot 6 \times 10^4$ C.G.S. units, is in fairly good agreement with Taylor's mean value, and the agreement is sufficient to show that κ is the same both for heat and for momentum.

This theory of eddy conductivity has been applied by Taylor in order to furnish an explanation of the diurnal variation of the velocity of the wind at the surface and in the lower layers of the atmosphere. Two important conclusions have been reached. He has shown† that when once the steady state has been reached, a state which previous theories claiming to explain this diurnal variation took no account of, a relation can be found between the undisturbed wind (i.e. the geostrophic wind), the surface wind, and the angle between the direction of the isobars and that of the surface wind. This relation takes the form

$$W/G = \cos\alpha - \sin\alpha, \qquad \text{(6)}$$

where W represents the surface wind, G the geostrophic wind, and α the angle between their directions. The accuracy of this relation has been tested by comparing values of α observed by G. M. B. Dobson‡ with the calculated values for certain winds over Salisbury Plain. Some of these results are given in Table IV.

TABLE IV.

$$W/G = \cos\alpha - \sin\alpha.$$

	Light Winds.	Moderate Winds.	Strong Winds.
Observed value of W/G	0·72	0·65	0·61
α observed	13 deg.	$21\frac{1}{2}$ deg.	20 deg.
α calculated	14 ,,	18 ,,	20 ,,

* "Recherches sur les courants les plus bas de l'atmosphère au-dessus de Paris", *Upsala Soc. Scient. Acta.*, 2 (Ser. 4), 1908, No. 2.

† "Eddy Motion in the Atmosphere", *Phil. Trans.* A, **215** (1915). See note, p. 285.

‡ *Quar. Jour. Roy. Met. Soc.*, **40**, p. 123 (1914).

The table shows that the agreement between observed and calculated values is very close; with α greater than 45°, the equation, however, no longer holds.

The other conclusion, as shown by Taylor,* on the assumption that the lag in the variation in wind velocity behind the variation n turbulence which gives rise to it is small, is that the daily variation in turbulence is sufficient to explain qualitatively, and to a certain extent quantitatively, the characteristics of the daily variation in the wind velocity. If the geostrophic wind G be reduced by surface friction so that the direction of the surface wind is inclined at an angle α to the "undisturbed" wind, then it is found that the force of the surface friction, or the rate of loss of momentum to the surface, is given by $2\kappa\rho BG \sin\alpha$, where $B = \sqrt{\omega \sin\phi/\kappa}$. As before, ω is the angular velocity of the earth and ϕ the latitude. The relation between this force of friction F and the velocity of the surface wind has also been examined by Taylor,† and found to be

$$F = 0{\cdot}0023\rho W^2.$$
$$\therefore\ 0{\cdot}0023\ W^2 = 2\kappa BG \sin\alpha.$$

If now numerical values be given to ω and ϕ in $\kappa = \omega\sin\phi/B^2$ we find that

$$\frac{1}{BG} = \frac{20{\cdot}4}{\sin\alpha}(\cos\alpha - \sin\alpha)^2, \ldots\ldots\ldots\ldots (7)$$

for $\omega = 0{\cdot}000073$, and $\sin\lambda = 0{\cdot}77$, since for Salisbury Plain $\lambda = 50°$ N. The values of $1/BG$ can therefore be found for a series of values of α. Also from the same equation we see that κ/G^2 is a function of α. If we tabulate the values of κ/G^2 for the same series of values of α, we can find then the relations between α and κ. These various values are given in Table V (p. 276).

Basing his discussion upon these values of the constants, Taylor has constructed the curves given in fig. 8. The abscissæ represent the ratio of the wind velocity at any height to the geostrophic wind, while the ordinates give the ratio of the height to the geostrophic wind. If the geostrophic wind be 10 m. per second, then the numbers for the ordinates will give the heights in dekametres, and those for the abscissæ the velocities in dekametres per second. The shape of each curve is determined by the value of α chosen, each curve having its α value attached to it. Consequently where

* *Proc. Roy. Soc.* A, **94**, p. 137 (1917).
† *Proc. Roy. Soc.* A, **92**, p. 198 (1916).

TABLE V

α, Degrees.	1/BG, C.G.S. Units.	κ/G^2, C.G.S. Units.
4	252	3·54
6	155	1·35
8	106	0·635
10	77·5	0·338
12	58·5	0·192
14	44·8	0·116
16	34·9	0·069
18	27·3	0·042
20	21·9	0·027
22	16·7	0·0156
24	12·9	0·0094
26	9·9	0·0055
28	7·4	0·0031
30	5·5	0·0017
32	3·7	0·00085
34	2·6	0·00038
36	1·7	0·00016

the geostrophic wind and the deviation at the surface are known, the curves enable us to determine the wind velocity at any desired height.

The curves may also be used to find the variation in velocity at a particular height under varying conditions of κ. The value of κ for the open sea, for Salisbury Plain, and for Paris we saw to be 3×10^3, 5×10^4, and 10×10^4 C.G.S. units respectively. If we take $\alpha = 10°$, then $\kappa/G^2 = 0{\cdot}338$, which means that under these three conditions G must have the values 0·9, 3·8, and 5·4 m. per second respectively. Therefore the same geostrophic wind will suit different curves if the value of κ be altered, which it will be according to the exposure of the station, the season of the year, and the time of day.

From the foregoing we see how the wind at the surface and in the lowest layers differs considerably from the geostrophic value. As we ascend above the surface, a nearer approach is made to the geostrophic values, for the effect of surface friction diminishes with height. Turbulence also diminishes as we ascend, its influence being on an average very little felt at 1000 m., though on occasions it may reach to 2000 m.

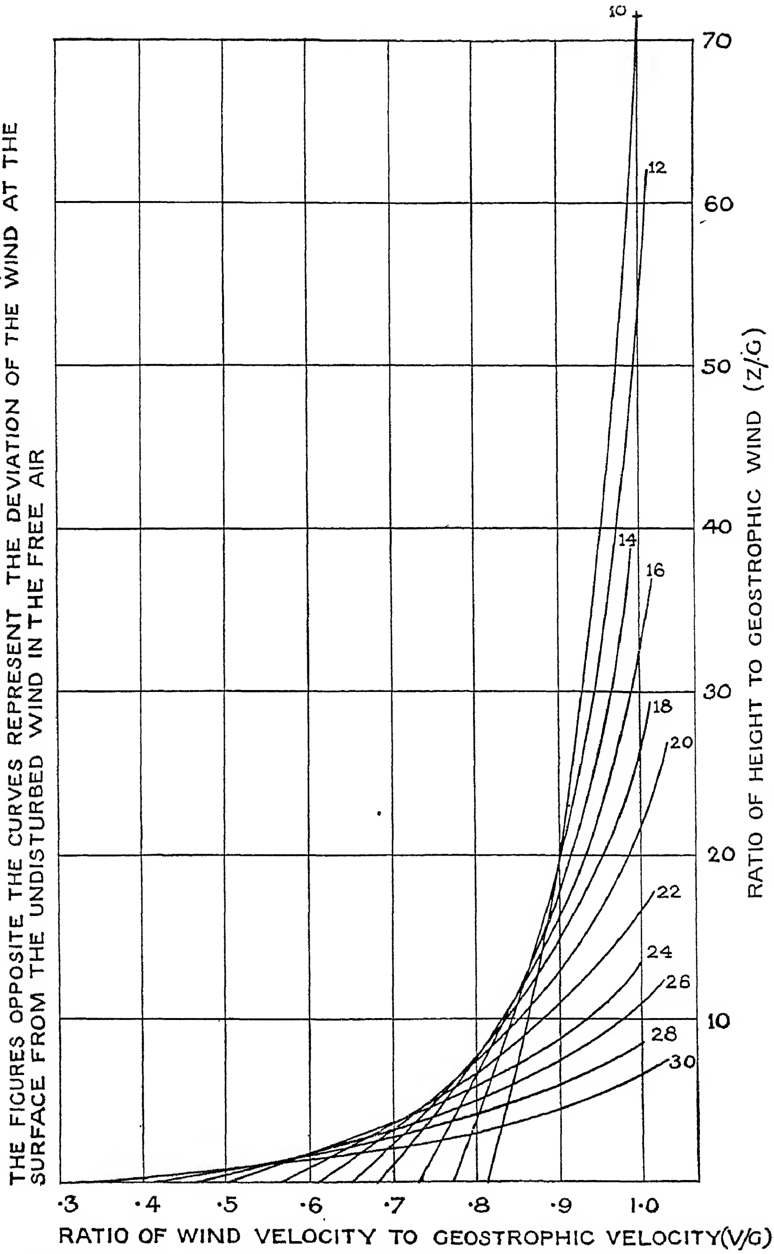

Fig. 8.—Curves showing Variation of Wind Velocity with Height, according to the Theory of the Diffusion of Eddy-motion. (Taylor)

The spiral of turbulence affords another method of representing the variation with height of wind velocity in magnitude and direction. In this method, first introduced by Hesselberg and Sverdrup * in 1915, when lines representing the wind velocity are drawn from the point at which the wind is measured, then their extremities lie on an equiangular spiral having its pole at the extremity of the line which represents the geostrophic wind. Thus in fig. 8*a*, if O be taken as the origin and O*g*X the direction of the *x*-axis, O*g* represents the geostrophic wind G, OS the surface wind and $\angle$ SO*g* the angle α between the two. The wind at any height Z is represented by OP, and is the resultant of the geostrophic wind G and of another component represented by *g*P of magnitude $\sqrt{2}\mathrm{G}\sin\alpha e^{-\mathrm{B}z}$ and acting in a direction which makes an angle $(\alpha + \frac{3\pi}{} - \mathrm{B}z)$ with the

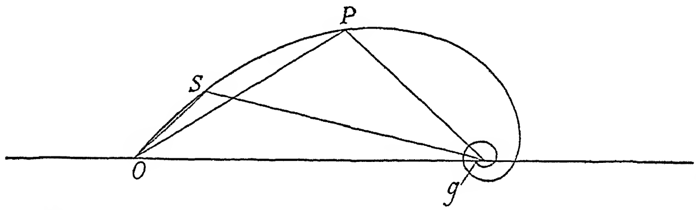

Fig. 8*a*.—Equiangular spiral representing velocity and direction of wind at any height

geostrophic wind. B has the same meaning as previously. An analysis of this method has been given by Brunt,† on the assumption that the coefficient κ is constant and that the geostrophic wind is the same at all levels. In a note added to the paper, Brunt also deals with the case where κ varies inversely as the height or inversely as the square of the height. The problem of κ varying as a linear function of the height has been considered by S. Takaya ‡ in a paper "On the coefficient of eddy-viscosity in the lower atmosphere". The solution enables the components of the wind to be calculated. The relations are equivalent to those found by Taylor (see Note 1). It must not be concluded, though the mathematical analysis appears to indicate it, that whenever a test is made on the wind that the results will produce an equiangular spiral. The gustiness of the wind prevents this, so that only when the mean of a large number of ascents is dealt with may one expect the wind values to form the equiangular spiral.

* "Die Reibung in der Atmosphare." *Veroff. d. Geophys. Inst. d. Univ. Leipzig*, Heft 10, 1915. † *Q. J. Roy. Meteor. Soc.*, 46, 1920, p. 175.

‡ *Memoirs of the Imperial Marine Observatory, Kobe, Japan*, Vol. IV, No. 1, 1930.

The next region to be considered stretches from the surface to a height of approximately 8000 m.

Observations with pilot balloons indicate that the geostrophic velocity is reached on an average below 500 m., while the direction is not attained until about 800 m. above the ground.* Each quadrant shows its own peculiarities, however. Thus Dobson† finds that for north-east winds the gradient velocity is reached at 915 m., for south-east below 300 m., for south-west about 500 m., and for

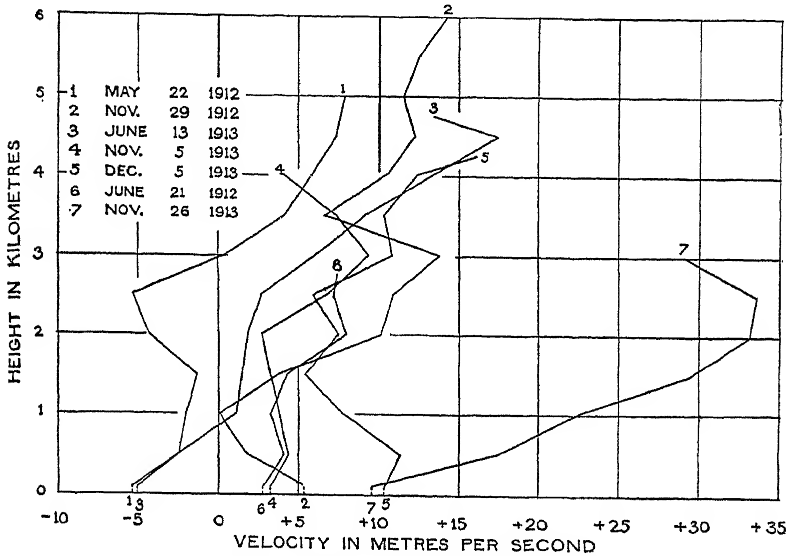

Fig. 9.—W. to E. Component of Wind Velocity on East Coast

north-west below 300 m. Also the winds in the north-east and south-east quadrants show little or no increase after reaching the geostrophic value, those in the north-east often showing a decrease, while those in south-west and north-west quadrants are marked by a continual increase beyond the geostrophic value, the velocity in the north-west being at 2500 m., 145 per cent of this value. The variation in direction also differs according to the quadrant. In the north-east quadrant, even at 2500 m., Dobson finds that the direction is 6° short of the direction of the isobars. On the other hand, in the south-east quadrant the direction of the isobars is reached at 600 m., and above this level the wind veers still farther. The south-west winds behave somewhat similarly, only the gradient

* For theoretical treatment, see Note II, p. 285*a*.

† *Q. J. R. Met. Soc.*, 40, p. 123 (1914).

direction is not attained until 800 m. is reached, while in the north-west quadrant the direction follows the isobars at 600 m. In this last quadrant, however, no further veer occurs until 1200 m. is reached, when a further veer begins. On the average the deviation of the surface wind from the gradient decreases from north-east to north-west, passing clockwise, Dobson's mean values being 27°, 24°, 19°, and 11° respectively.

These results refer to an inland station. When we come to

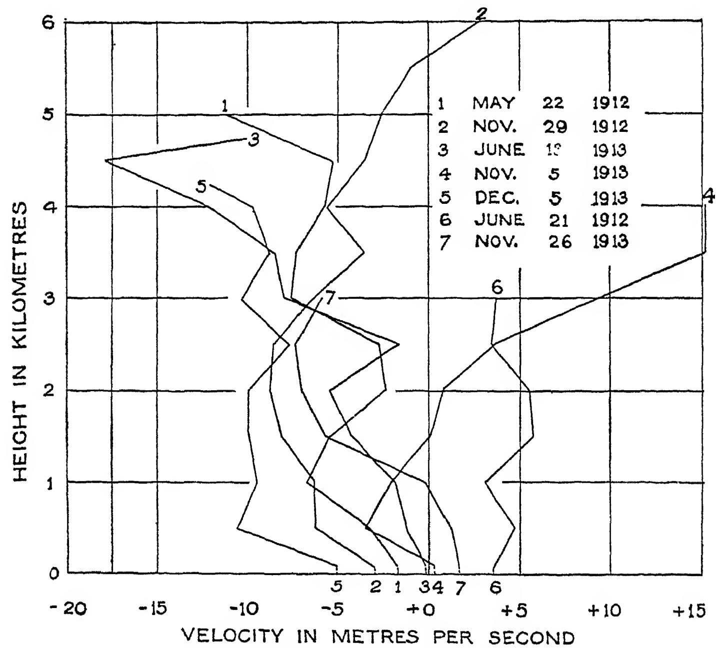

Fig. 10.—S. to N. Component of Wind Velocity on East Coast

deal with a station on the coast we find even greater complications. Figs. 9 and 10, which represent the component velocities of a number of observations carried out with the aid of two theodolites at Aberdeen by the author,* serve to show the irregularities of these velocities. A greater variation is shown in the west-east component than in the south-north, as is to be expected from the exposure of the station. On the whole, the west-east component shows a tendency to increase with height, while any east-west velocity gradually dies out. The south-north diagram gives mainly negative values,

* *Q. J. R. Met. Soc.*, **41**, p. 123 (1915).

i.e. the winds observed had mainly a north-south component. This component changes comparatively little in the first 4000 m., though higher up there is a tendency to increase indicated both for south-north and north-south winds. This is in general agreement with the results arrived at by Dobson.

Cave, in his *Structure of the Atmosphere in Clear Weather*, has given the results of observations carried out at Ditcham Park. When we consider his results for heights between 2500 m. and 7500 m., we find that there is a decided increase with height in the westerly components, the easterly components tending to die out. On the other hand, both southerly and northerly components show an increase, the range at 7500 m. being much greater than at 2500 m., the actual values being from −20 m. per second to +20 m. per second at the higher level, to −6 m. per second to +9 m. per second at the lower.

As the result of investigation, Cave has divided his soundings in the troposphere * into five different groups, and has added a sixth for winds in the stratosphere.† These are:

(*a*) 1. " Solid " current; little change in velocity or direction.
2. No current up to great heights.
(*b*) Considerable increase in velocity.
(*c*) Decrease of velocity in the upper layers.
(*d*) Reversals or great changes in direction.
(*e*) Upper wind blowing outward from centres of low pressure: frequently reversals at a lower layer.
(*f*) Winds in the stratosphere.

In group (*a*) the gradient direction and velocity are reached early, and thereafter the wind remains nearly constant. There is practically no temperature gradient, and the pressure distribution at different heights is similar to that at the surface.

Group (*b*) is mainly due to a westerly or south-westerly type, and represents the average conditions where depressions are passing eastwards over the British Isles. There is here a marked temperature gradient over the area.

In group (*c*) are included mainly easterly winds, the pressure

* Troposphere, i.e. the part of the atmosphere in which the temperature falls off with increasing altitude. In latitude 55° it extends from the surface up to about 11 Km.; in the tropics it extends to about 17 Km.

† Stratosphere, i.e. the external layer of the atmosphere in which there is no convection. It lies on the top of the troposphere, and the height of its base above the surface varies from equator to poles (see troposphere). The temperature changes within it are in a horizontal direction.

distribution showing an anticyclone to the north. The gradient velocity is reached at about 500 m., the gradient direction at a point a little higher. Thereafter decrease in velocity takes place and occasionally a *backing* of the wind, though the latter does not invariably occur.

With " reversals ", which are placed in group (*d*), the surface wind is almost always easterly; the upper, westerly or south-westerly.

Here we have a warm current passing over a colder, and the result is generally rain. Very often in summer there is found a south-west current passing over a south-east, the two being associated with shallow thunderstorm depressions, the south-west current supplying the moisture to form the cumulo-nimbus clouds.

From an examination of the winds in group (*e*), it is almost always found that the depressions from which the winds come advance in the direction of the upper air current. This is particularly the case with north-westerly upper winds. With south-westerly upper winds we have very often conditions similar to those mentioned under (*d*), with corresponding results.

Observations within the stratosphere are comparatively few, but, in general, they show that the wind within this region tends to fall off with increase in height, and that the direction is almost invariably from some point on the west side of the north-south line.

Several models, which show at a glance how the air currents change with height, have been constructed by Cave. For a description of these and a full account of his investigation the reader is referred to his book already mentioned.

Let us now examine the wind structure in these upper regions of the atmosphere from the theoretical standpoint.

We have already noted that the variations in the distribution of pressure in the upper atmosphere are closely correlated with the variations in the temperature distribution. Starting with the ordinary equation for the diminution of pressure with height and combining it with the characteristic equation for a permanent gas, we are able to find an equation giving the variation of pressure gradient with height. These equations are:

$$\frac{\partial p}{\partial z} = -g\rho, \quad \ldots\ldots\ldots\ldots\ldots\ldots (8)$$

and $$p/\mathrm{T} = \mathrm{R}\rho,$$

the latter giving $$\frac{\partial \rho}{\rho} = \frac{\partial p}{p} - \frac{\partial \mathrm{T}}{\mathrm{T}}. \quad \ldots\ldots\ldots\ldots\ldots\ldots (9)$$

Also if the horizontal pressure and temperature gradients be written as $\frac{\partial p}{\partial x} = s$, and $\frac{\partial T}{\partial x} = q$, respectively, then we have

$$\frac{\partial s}{\partial z} = \frac{\partial^2 p}{\partial x \partial z},$$

i.e. from equation (8)

$$\frac{\partial s}{\partial z} = -g\frac{\partial \rho}{\partial x}. \qquad \text{(10)}$$

Therefore combining (9) and (10) we have for the change with height in pressure gradient,

$$\frac{\partial s}{\partial z} = -g\rho\left\{\frac{1}{p}\frac{\partial p}{\partial x} - \frac{1}{T}\frac{\partial T}{\partial x}\right\}$$

$$= g\rho\left\{\frac{q}{T} - \frac{s}{p}\right\} \qquad \text{(11)}$$

To find numerical values we must substitute for ρ its value p/RT. For dry air R $= 2{\cdot}869 \times 10^6$ C.G.S. units, while for air saturated with water vapour at $273a$ its value is, $2{\cdot}876 \times 10^6$ C.G.S. units. This differs only slightly from the value for dry air. Also the uncertainties which arise in connection with the determination of the wind velocity in the upper air are greater than the variations in R, and therefore the value for dry air may be used on all occasions without any appreciable error. With this value, and with g as 981 cm./sec.^{2}, we have

$$\frac{\partial s}{\partial z} = 3{\cdot}42 \times 10^{-4}\frac{p}{T}\left\{\frac{q}{T} - \frac{s}{p}\right\}. \qquad \text{(11a)}$$

If now we express the variation in pressure in millibars per metre of height and take the gradients in pressure and temperature over 100 Km., the rate of increase of pressure gradient per metre of height in millibars per 100 Km. is

$$3{\cdot}42 \times 10^{-2}\frac{P}{T}\left\{\frac{Q}{T} - \frac{S}{P}\right\},$$

P and S being expressed in millibars, T and Q in degrees absolute.

The variation in pressure gradient depends therefore on the difference of the quantities Q/T and S/P. Now T falls from about $280a$ at the surface to approximately $220a$ at 9 Km., whereas P changes from 1010 millibars to nearly 300 millibars within the

same range. We see, therefore, that S/P runs through a considerable range of values, while Q/T remains comparatively constant. The variation in pressure difference is therefore not constant within the region considered, but is likely to show positive values at first, then change through zero to negative values higher up. If the pressure difference remained constant up to 9 Km., then $V\rho$ would be constant, and the velocity of the wind would increase in inverse proportion to the density of the air as we ascend. Now Egnell believed that he found by observation of clouds that $V\rho$ actually was constant, and in consequence this law, $V\rho$ = a constant, has been termed Egnell's Law. We have seen, however, that the observations by pilot balloons do not confirm the law. The wind very often shows an increase in velocity with increase in height, especially winds with a westerly component, but this increase is generally less, even in the latter case, than in accordance with a uniform gradient. Equation (11) is therefore much more in agreement with the behaviour of the actual winds than a constant pressure gradient would be.

The variation of wind with height can now be obtained by combining equation (11) with the relation

$$s = 2v\rho\omega \sin\phi, \text{ or, } v\rho = s/2\omega \sin\phi \quad \ldots\ldots\ldots(12)$$

Let v be the component of the wind velocity parallel to the y-axis drawn towards the north, the x-axis being drawn towards the east. Then

$$\rho\frac{\partial v}{\partial z} + v\frac{\partial \rho}{\partial z} = \frac{\partial s}{\partial z}/2\omega \sin\phi,$$

$$\text{i.e. } \frac{1}{v}\frac{\partial v}{\partial z} + \frac{1}{\rho}\frac{\partial \rho}{\partial z} = \frac{1}{s}\frac{\partial s}{\partial z},$$

$$\begin{aligned}
\text{or} \qquad \frac{1}{v}\frac{\partial v}{\partial z} &= \frac{1}{s}\frac{\partial s}{\partial z} - \frac{1}{\rho}\frac{\partial \rho}{\partial z} = -\frac{g}{s}\frac{\partial \rho}{\partial x} - \frac{1}{\rho}\frac{\partial \rho}{\partial z} \\
&= -\frac{g\rho}{s}\left(\frac{1}{p}\frac{\partial p}{\partial x} - \frac{1}{T}\frac{\partial T}{\partial x}\right) - \left(\frac{1}{p}\frac{\partial p}{\partial z} - \frac{1}{T}\frac{\partial T}{\partial z}\right) \\
&= -\frac{g\rho}{s}\left(\frac{s}{p} - \frac{q}{T}\right) + \frac{g\rho}{p} + \frac{1}{T}\frac{\partial T}{\partial z} \\
&= \frac{1}{T}\left\{\frac{g\rho q}{s} + \frac{\partial T}{\partial z}\right\},
\end{aligned}$$

$$\text{i.e. } \frac{s}{v}\frac{\partial v}{\partial z} = \frac{1}{T}\left(g\rho q + s\frac{\partial T}{\partial z}\right);$$

and as $s/v = \rho 2\omega \sin\phi$,

$$\therefore \frac{\partial v}{\partial z} = \frac{1}{\rho T 2\omega \sin\phi}\left\{-q\frac{\partial p}{\partial z} + s\frac{\partial T}{dz}\right\}$$

$$= \frac{1}{2\omega \sin\phi\rho T}\left\{\frac{\partial p}{\partial x}\frac{\partial T}{\partial z} - \frac{\partial p}{\partial z}\frac{\partial T}{\partial x}\right\} \ldots\ldots (13)$$

Along the x-axis the corresponding value will be

$$\frac{\partial u}{\partial z} = -\frac{1}{2\omega \sin\phi\rho T}\left\{\frac{\partial p}{dy}\frac{\partial T}{dz} - \frac{\partial p}{\partial z}\frac{\partial T}{\partial y}\right\} \ldots\ldots (13a)$$

Here the negative sign must be used because if the pressure increase towards the north then the wind will be from the *east*.

If then the wind be observed at various levels it is possible from these equations to calculate the separation of the isobars and isotherms at the different levels. For this purpose it is necessary to know the values of p and T for each level considered. In any particular case the normal values of these quantities for the month in which the observation takes place may be taken without any very serious error. We may then proceed to calculate the separation of the isobars and isotherms at intervals of a kilometre in the following way.

The change of pressure difference we already expressed in the form

$$\frac{\partial s}{\partial z} = 3{\cdot}42 \times 10^{-4}\frac{p}{T}\left\{\frac{q}{T} - \frac{s}{p}\right\} \text{ in C.G.S. units.}$$

If then the pressure be expressed in millibars and the temperature in degrees absolute, the change of pressure-difference per kilometre of height may be written as

$$\Delta s = 34{\cdot}2\frac{P}{T}\left(\frac{\Delta T}{T} - \frac{\Delta P}{P}\right), \ldots\ldots\ldots\ldots (14)$$

where ΔP and ΔT are the horizontal changes per 100 Km. in pressure and temperature respectively. The wind velocity W due to this pressure difference ΔP can be found from the relation

$$W = \frac{R}{2\omega \sin\phi}\frac{T}{P}\Delta P = K\frac{T}{P}\Delta P \ldots\ldots\ldots\ldots (15)$$

If U and V be the components of W from west to east and from south to north respectively, then the components of pressure

difference at any level as deduced from the wind observations are

$$\left.\begin{aligned} \Delta_N P &= \frac{1}{K}\frac{P}{T}U \\ \text{and} \qquad \Delta_W P &= \frac{1}{K}\frac{P}{T}V \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (15a)$$

Similarly the components of temperature difference can be expressed from equation (14) in the form

$$\left.\begin{aligned} \Delta_N &= T\frac{T}{P}\left(\frac{\Delta s}{34\cdot 2} \times T + \Delta_N P\right) \\ \text{and} \qquad \Delta_W &= T\frac{T}{P}\left(\frac{\Delta s}{34\cdot 2} \times T + \Delta_W P\right) \end{aligned}\right\} \quad \ldots\ldots\ldots (14a)$$

Table VI (p. 287) is an example of the application of these equations. Of the last four columns the first two give the separation in kilometres between the component isobars where the difference is 1 millibar, the second two between the component isotherms where the difference is 1° C. When the direction of the resultant isobars and isotherms for the various levels are calculated we find the following directions:

Height in Km.	0	1	2	3	4
Isobar from	270°	272°	271°	214°	263°
Isotherm from	—	15°	271°	210°	334°

This appears to indicate the approach of a warmer current from the south-west at a height over 3 Km. The pressure distribution at 7 hr. on the 28th gave a depression over Iceland with a surface temperature of 50° F. The increase of temperature indicated at the 3000-m. level is apparently due therefore to the warm air from this depression pushing its way across the colder northerly current. This seems to be in agreement with Bjerknes' theory* of the circulation of air within a cyclone.

The observations we have been considering hitherto refer to one station only, so that we have obtained only a very small section of the isobars and isotherms for the different levels. If a number of observations be made simultaneously at different stations over the British

* *Q. J. Roy. Met. Soc.*, 46, p. 119.

TABLE VI

Computation of Pressure and Temperature Distribution from Pilot Balloon Ascent of 28th May, 1914

Height in Km.	Vel., m.p.s.	Direction.	W.-E. Component.	S.-N. Component.	$\Delta_N P$	$\Delta_W P$	$\Delta_N T$	$\Delta_W T$	$\frac{100}{\Delta_N P}$	$\frac{100}{\Delta_W P}$	$\frac{100}{\Delta_N T}$	$\frac{100}{\Delta_W T}$
0	0·0	..	− 0·0	+ 0·0	+ 0·60	0·0	..	..	+ 167	..	..	..
1	1·7	291°	+ 1·6	− 0·6	+ 0·22	− 0·8	− 0·89	+ 0·23	+ 455	− 1250	− 112	− 435
2	2·9	288°	+ 2·8	− 0·9	+ 0·35	− 0·10	+ 0·48	− 0·09	+ 286	− 1000	+ 208	− 1111
3	7·0	231°	+ 5·3	+ 4·4	+ 0·59	+ 0·49	+ 0·95	+ 1·96	+ 169	+ 204	+ 103	+ 51
4	8·9	263°	+ 8·7	+ 1·1	+ 0·87	+ 0·11	+ 1·32	− 1·18	+ 115	+ 909	+ 76	− 85

Isles, say, then a series of maps may be drawn showing the air flow at each level. These will afford an indication of the distribution of pressure and temperature at the various levels. In fig. 11 the pressure distribution at the surface at 18 hr. on 7th September, 1922, is given in (*a*). The following four members of the series show approximately the run of the isobars at the levels indicated as deduced from pilot observations made at 17 hr., while the last of the series depicts the pressure distribution at the surface at 7 hr. on the following morning. The separation of the isobars is 2 millibars in every case. At the surface the isobars run from north-east to south-west, but higher up the direction changes towards a north to south direction. This appears to indicate a mass of rather warmer air towards the west or south-west, especially about the 6000-ft. level. The velocities, however, are comparatively small and therefore a break-up of the system is not to be expected. Instead, as the 7 hr. chart of the following morning shows, there has taken place a further development, and the direction of the isobars at the surface has now become much more in accordance with the upper air isobars of the previous evening.

We must now consider the case of curved isobars. In the expression for the gradient wind determined near the beginning of our survey there were found to be two parts, one dependent upon the rotation of the earth, the other on the curvature of the path. Hitherto we have dealt only with the first part, but now we shall consider briefly the effect of the curvature of the path upon the relation of the wind to the distribution of pressure. In discussing the circulation of air in temperate latitudes, Shaw arrives at the following conclusion.* "Thus out of the kaleidoscopic features of the circulation of air in temperate latitudes two definite states sort themselves, each having its own stability. The first represents air moving like a portion of a belt round an axis through the earth's centre. It is dependent upon the earth's spin, and the geostrophic component of the gradient is the important feature; the curvature of the isobars is of small importance. The second represents air rotating round a point not very far away: it is dependent upon the local spin, and the curvature of the isobars with the corresponding cyclostrophic component of the gradient is the dominant consideration."

Up to the present we have been considering only one point in the path of the air, and the lines of flow of the air at that point we have regarded as coincident with the isobars in the upper air and making a definite angle with them near the surface owing to the

* *Manual of Meteorology*, Part IV, p. 236.

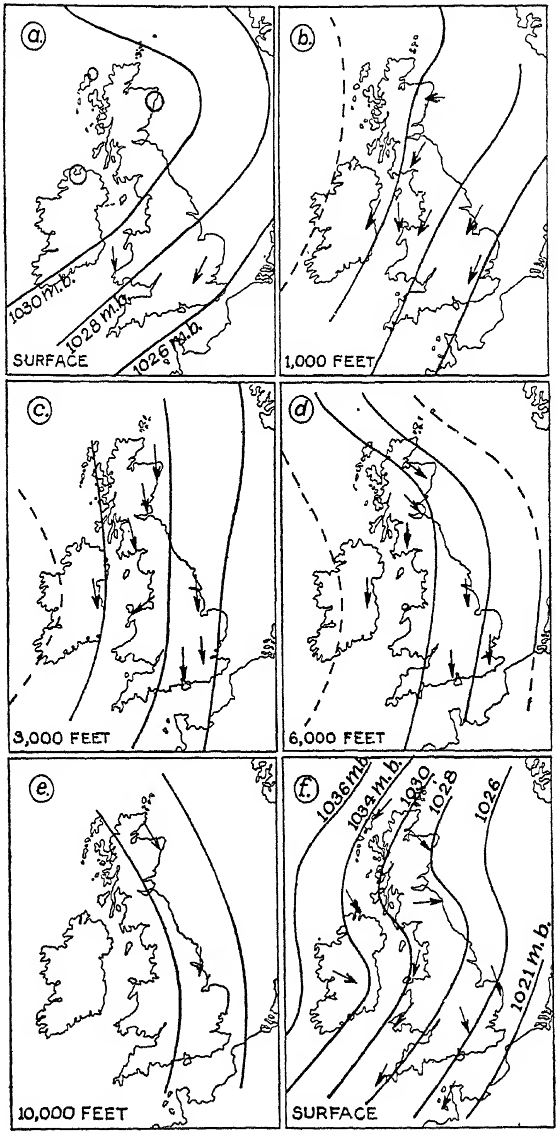

Fig. 11.—Map showing the Pressure Distribution and Wind Direction at the Surface at 18 hr. on 7th September (*a*); at 7 hr. on 8th September (*f*); and (*b* to *e*) the approximate Direction of the Isobars at 1000 ft., 3000 ft., 6000 ft., and 10,000 ft., as deduced from Pilot Balloon Observations at 17 hr. on 7th September, 1922.

turbulence in the atmosphere. When we come to consider a succession of states, however, we see that the paths of the air are not necessarily coincident with the lines of flow or the latter with the isobars. In the case of the first state mentioned above by Shaw, the isobars are straight and the paths of the air are coincident with the lines of flow; but when the two states are superposed and a series of maps drawn giving the pressure distribution at definite intervals, it is seen that the paths of the air are no longer coincident with the lines of flow or with the isobars. What then are the paths of air in a cyclone?

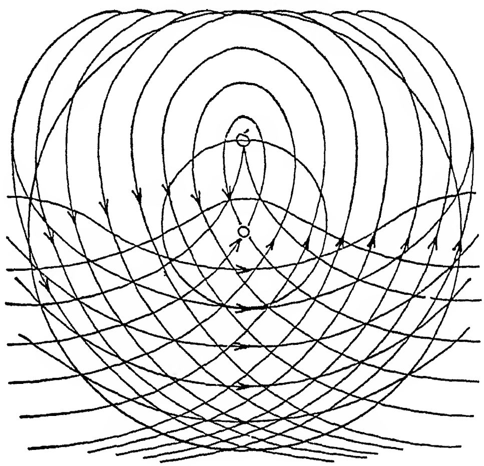

Fig. 12.—Trajectories of Air for a Normal Cyclone (from Shaw's *Manual of Meteorology*)

A partial solution of the problem may be reached after the following manner. It is a well-known fact of experience that one of the characteristics of a cyclone is that it travels across the map, and when the isobars are circular that the velocity of translation is rapid. We shall here confine ourselves therefore to the examination of a circular, rapidly moving storm, termed the "normal" cyclone or cartwheel depression.

If two horizontal plane sections of a normal cyclone be taken, including between them a thin lamina or disc of rotating air, and if this disc travel unchanged in a horizontal direction, then to obtain the actual velocities of any point on the disc the velocity of translation must be combined with the velocity of rotation. If the velocity of translation be V and the angular velocity n, then the centre of instantaneous rotation will be distant from the centre of the disc a distance V/n. This centre of instantaneous rotation will travel in a line parallel to the line of motion of the centre of the disc. The actual paths of the air particles are traced out by points attached to a circle which rolls along the line of instantaneous centres and whose radius $= V/n$. Fig. 12 represents these trajectories. The figure shows, in one position, the circle of radius V/n, its centre O, and the instantaneous centre O′. The circle rolls on

the line through O′ perpendicular to OO′. The path of a particle has a cusp, a loop, or neither, according as the tracing point is on, without, or within the circle.

From this we see that in the normal cyclone there are two centres, the one, O (fig. 12), the actual centre of the rotating disc which is termed the *tornado* centre, the other, O′, the centre of instantaneous rotation termed the *kinematic* centre. They lie on a line perpendicular to the path of the cyclone, and are distant from each other by the length V/n.

In such a system as this the isobars will not coincide exactly with the lines of flow. For the present neglecting the variation in latitude and in density, and also the curvature of the earth's surface, we shall regard the cyclone as moving along a horizontal plane. In that case the system of isobars will be obtained by compounding a system of circular isobars embedded in a field of straight isobars. The centre of the circular system will not coincide with either of the centres already referred to, but will be at a distance from the kinematic centre $= V/(2\omega \sin\phi + n)$ and lie on the line joining the kinematic and tornado centres. This centre has been termed the *dynamic* centre, and is the centre of the isobaric system as drawn on a map. It is therefore quite easily identified, but one must bear in mind that it is not the *only* centre in a normal cyclone.

If we take the centre of the rotating disc as origin, with x and y axes towards the east and north respectively, then for an eastward velocity of translation V, the pressure will diminish uniformly towards the north at the rate $2\rho V\omega \sin\phi$, i.e. the field of pressure will be represented by

$$\int_{p'_0}^{p'} dp = -\int_0^y 2\rho V\omega \sin\phi\, dy,$$

where p' = pressure at any point, and p'_0 = pressure at any point on the x-axis;

$$\text{i.e. } p' - p'_0 = -2\rho V\omega \sin\phi y \qquad \text{(16)}$$

For the circular field with its centre at the origin we have

$$\int_{p_0}^{p} dp = \int_0^r (2\rho v\omega \sin\phi + v^2\rho \cot\alpha/R)\, dr,$$

where p is the pressure at any point distant r from the origin, and p_0 the pressure at the origin.

If we neglect the curvature of the earth, then $v^2 \cot\alpha/R = v^2/r$. Also $v = rn$.

$$\therefore \int_{p_0}^{p} dp = \rho n \int_0^r (2\omega \sin\phi + n)\, r dr,$$

i.e. $$p - p_0 = \rho\frac{n}{2}(2\omega \sin\phi + n)\, r^2 = \rho\frac{n}{2}(2\omega \sin\phi + n)(x^2 + y^2) \quad (17)$$

By combining the two equations (16) and (17), we have for the resultant field

$$P - P_0 = \frac{\rho n}{2}(2\omega \sin\phi + n)(x^2 + y^2) - 2\rho V\omega \sin\phi y \ldots (18)$$

This represents a circular field of pressure round a point

$$x = 0,\ y = \frac{2\omega \sin\phi V}{n(2\omega \sin\phi + n)},$$

and P_0 is the pressure at the centre of this field and not at the origin.

Now the distance of the kinematic centre from the tornado centre which was chosen as origin is equal to V/n. Therefore the distance of the kinematic centre from the dynamic centre is

$$V/n - \frac{2\omega \sin\phi}{2\omega \sin\phi + n} \times \frac{V}{n} = V/(2\omega \sin\phi + n).$$

We see, therefore, that this combination of a field of straight isobars with a circular system embedded in it is sufficient to give the field of pressure necessary to keep the disc rotating.

In the normal cyclone it follows that the centre of low pressure being not the centre of the lines of flow, the wind possesses a definite counter-clockwise velocity at the centre of low pressure. When an actual example of a rapidly moving circular storm such as that of 10th–11th September, 1903, is examined, we find that the system actually does possess such a wind, and consequently this divergent wind, which has often been regarded as accidental, is in reality in perfect agreement with the pressure system. Another feature which the normal cyclone possesses in common with an actual circular cyclone is the greater incurvature in the rear of the cyclone as compared with that in front.

" From these considerations," says Shaw,* " we are led to accept the conclusion to be drawn from the conditions of the normal cyclone,

* *Manual of Meteorology*, Part IV, p. 245

namely, that the wind *calculated from the gradient by the full formula using the curvature of the isobars, gives the true wind in the free air not at the point at which the gradient is taken but at a point distant from it along a line at right angles to the path and on the left of it by the amount* $V/(2w \sin\phi + n)$."

The calculated trajectories in the case of the normal cyclone

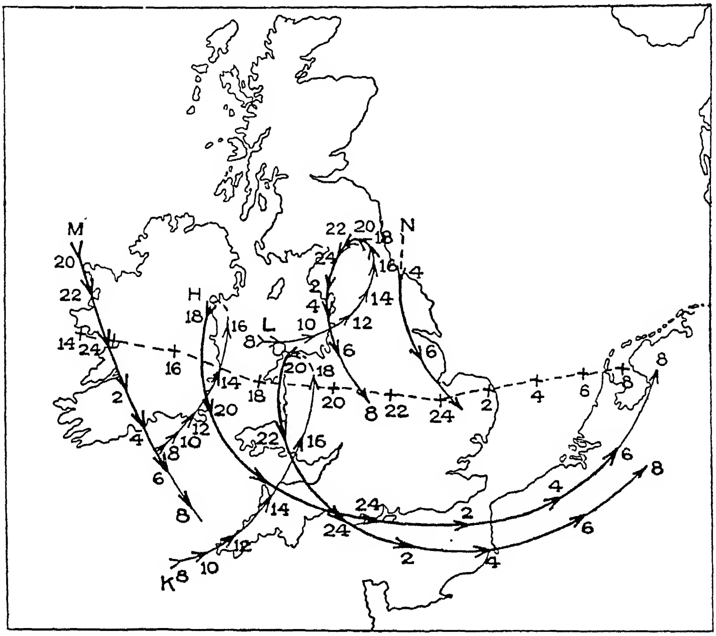

Fig. 13.—Trajectories of Air in Circular Storm, 10th to 11th September, 1903

have already been referred to and given in fig. 12. A comparison of these with the actual trajectories for the storm of 10th–11th September, 1903 (fig. 13), shows at once the remarkable similarity between the two sets, indicating still further that this actual cyclone and the normal cyclone are very close akin the one to the other. The trajectories of the September cyclone are reproduced from *The Life History of Surface Air Currents.**

Here we have considered in a very fragmentary way only one form of stable rotation, namely the circular. Even then no account was taken of the discontinuity of velocity which must occur at the

* *Life History of Surface Air Currents,* by W. N. Shaw and R. G. K. Lempfert, M.O., No. 174, London, 1906.

edge of the rotating disc in the normal cyclone, but it is possible to show that this discontinuity can be accommodated by including in the revolving column of air an outer region represented by the law of the simple vortex with vr constant. The regions beyond what have hitherto been included in the revolving disc will also form part of the cyclone, therefore, and not simply belong to the environment.

For further treatment of this subject the reader is referred to treatises on dynamical meteorology, such as Shaw's *Manual of Meteorology*, as in this brief study some of the intricacies only of the problems of wind structure, rather than their solutions, have been placed before him.

Note I

The equations of motion of air over the surface of the earth when the effect of eddy viscosity is taken into account are, when the steady state has been reached and the motion is horizontal,

$$0 = -2\omega v \sin\phi + \kappa\frac{d^2u}{dz^2} \qquad \ldots\ldots(1)$$

$$0 = +2\omega u \sin\phi - 2\omega G \sin\phi + \kappa\frac{d^2v}{dz^2} \qquad \ldots\ldots(2)$$

On eliminating u from the above we find that the equation for v becomes

$$\frac{d^4v}{dz^4} + \frac{4\omega^2 \sin^2\phi}{\kappa^2}v = 0,$$

or

$$\frac{d^4v}{dz^4} + 4B^4v = 0, \text{ where } B^2 = \frac{\omega \sin\phi}{\kappa}.$$

Now v does not become infinite for infinite values of z, and the solution of the equation is therefore

$$v = A_2e^{-Bz}\sin Bz + A_4e^{-Bz}\cos Bz \qquad \ldots\ldots(3)$$

On differentiating this value of v twice with respect to z, and substituting in (2), we get

$$u = G + A_2e^{-Bz}\cos Bz - A_4e^{-Bz}\sin Bz \qquad \ldots\ldots(4)$$

Now G = value of the gradient wind velocity,
and therefore for great heights $u = G$, i.e. the gradient wind velocity, and $v = 0$.

The values of A_2, A_4 are found by imposing suitable boundary conditions.

Now at $z = 0$ these are

$$\left[\frac{du/dz}{u}\right]_{z=0} = \left[\frac{dv/dz}{v}\right]_{z=0}, \text{ and } \tan\alpha = -\left[\frac{v}{u}\right],$$

where α = angle between the observed wind and the gradient wind.

From these conditions

$$A_2 = \frac{-\tan\alpha(1+\tan\alpha)}{\tan^2\alpha+1}G; \quad A_4 = \frac{-\tan\alpha(1-\tan\alpha)}{\tan^2\alpha+1}G.$$

The surface wind $W = \sqrt{[u^2+v^2]_{z=0}}$

$$= \sqrt{A_4{}^2 + (A_2+G)^2}$$

$$= \frac{G}{1+\tan^2\alpha}\sqrt{\tan^2\alpha(1-\tan\alpha)^2+(1-\tan\alpha)^2}$$

$$= G\frac{(1-\tan\alpha)}{\sec\alpha} = G(\cos\alpha - \sin\alpha),$$

or $\quad W/G = (\cos\alpha - \sin\alpha)$.

Note II

The theory of eddy motion also accounts for the observed fact that the magnitude of the gradient wind is reached at a level lower than that at which the gradient direction is attained.

The height at which the gradient direction is reached is found from equation (3) of Note I by putting $v = 0$. If H_1 be the height, then

$$0 = A_2 \sin BH_1 + A_4 \cos BH_1,$$

and therefore $\quad \tan BH_1 = -\dfrac{A_4}{A_2}.$

Substitute the values of A_4 and A_2 already found, and we obtain

$$\tan BH_1 = -\frac{1-\tan\alpha}{1+\tan\alpha} = \tan\left(\alpha - \frac{\pi}{4}\right).$$

Since α is positive and less than $\frac{\pi}{4}$, the smallest value of H_1 is got from

$$BH_1 = \frac{3\pi}{4} + \alpha. \qquad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

The height H_2, at which the value of gradient velocity is reached, is given by $u^2 + v^2 = G^2$, which, on substitution, becomes

$$e^{-BH_2} = \frac{(1 + \tan\alpha)\cos BH_2 - (1 - \tan\alpha)\sin BH_2}{\tan\alpha}. \quad \ldots (2)$$

From this equation BH_2 can be found in terms of $\tan\alpha$.

The following table, given by Taylor, shows the values of BH_1, BH_2, and H_1/H_2 as α goes from 0° to 45°.

α.	BH_1.	BH_2.	H_1/H_2.
0	2·35	0·78	3·0
20	2·70	1·04	2·6
45	3·15	1·44	2·2

Now for Salisbury Plain Dobson found that the deviation α was, in a large number of cases, 20°; also that $\frac{H_1}{H_2}$ for this deviation was $\frac{800 \text{ metres}}{300 \text{ metres}} = 2{\cdot}66$.

This is in good agreement with the theoretical value 2·6.

CHAPTER IX

Submarine Signalling and the Transmission of Sound through Water

Although practically every other branch of science has had considerable technical application, that of acoustics has until the last few years remained practically in the academic stage, and few even among scientific men gave serious attention to it. Bells, gongs, whistles, sirens, and musical instruments have indeed been used from remote times both for enjoyment and for signalling purposes, but their development has mainly been on empirical lines, with but little assistance from the physicist.

The Great War has, however, brought about a striking change in this as in many other directions, and acoustics is now becoming not only an important branch of technology, but shows signs even of developing into the engineering stage and giving us a new and powerful method of power transmission, to judge by the pioneer work of M. Constantinesco, who has already developed it for the operation of rock drills and riveting machines, and shown how it may be applied to motors and other machines. Few branches of science now offer such possibilities to the inventor.

Acoustic signalling is of especial importance in connection with navigation, as sound is the only form of energy which can be transmitted through water without great loss by absorption. The relatively high electrical conductivity of water renders it almost opaque to light and to electromagnetic waves.

The present article deals principally with acoustic signalling under water, but certain allied problems, such as sound ranging, depth sounding, and other applications to navigation, will also be briefly referred to.

As is well known, sound consists of a vibratory disturbance of a material medium, such as a gas, solid, or liquid, and its phenomena

are almost of a purely mechanical nature. When a bell or tuning fork is struck it is thrown into vibration, and as any part moves forward it compresses the medium in front of it and also gives it a forward velocity. As the vibration reverses so that the movement is in the opposite direction, the mass or inertia of the medium keeps it moving forward, and a partial vacuum or rarefaction is produced, until the vibration again reverses and forms a fresh compression. These regions of compression and rarefaction therefore travel forward as a series of pulses or waves away from the source in much the same way as ripples are formed on the surface of a pond when a stone is dropped into it. In the case of the surface ripples, however, the real motion of the water is partly up and down, or transverse to the direction of movement of the waves, whereas in the case of sound the motion of each particle of the medium is mainly forwards and backwards along the line of propagation of the sound. Sound vibrations are therefore spoken of as longitudinal or in the direction of transmission, which differentiates them from all other kinds of vibrations, such as those of ordinary waves, light, or electromagnetic waves, which are said to be transverse. It at once follows from this that although many of the scientific principles of optics can be and indeed have been successfully applied to sound, there can be nothing in acoustics corresponding to polarization in light. This point is made clear at the outset, as the phenomena of light are fairly generally known and are most stimulating to acoustic development.

Fundamental Scientific Principles

In order to understand the operation of modern acoustic transmitting and receiving instruments properly, it will be well to start with a brief statement of certain scientific principles and definitions. Some of these are well known, but others require a few words of explanation.

Sounds are divided into musical notes and noises, and musical notes are spoken of as differentiated by intensity, pitch, and *timbre* or quality. A musical note is produced whenever the vibrations are of a regular character, so that each wave is similar to the previous one. The intensity or loudness of the note depends on the strength or amplitude of the vibration, its pitch on the number of vibrations per second or frequency, and its *timbre* or quality on the form of the vibration. The purest musical note is given by uniform

vibrations of a simple harmonic character, and if the wave-form is saw-toothed or shows any other variation from the sine form, the note is more or less piercing in quality, due to the presence of overtones or higher harmonics besides the fundamental pure tone.

Noises are differentiated from musical tones by having no regular character, and are made up of a number of vibrations of different intensity and pitch. Speech may be described as a noise from the point of view of acoustic transmission and reception, on account of the variable nature of the vibrations; and also the sound from machinery, ships, &c. This is a serious difficulty as regards the detection and recognition of such sounds, as nearly all transmitting and receiving devices are more or less "selective" in character, i.e. they respond better to certain definite frequencies and are relatively insensitive to others. Everyone knows that telephones or gramophones reproduce certain sounds better than others, and acoustic signalling, like wireless transmission, is far more effective with "tuned" devices, which are, however, very insensitive to other frequencies.

Velocity of Propagation

An accurate knowledge of the velocity of propagation of sound is of great importance in connection with acoustic signalling, especially as regards determination of range or position as in sound ranging. The velocity is very different in different substances, as it depends on the elasticity and density of the material, and therefore on its composition, pressure, and temperature. We are here concerned chiefly with the velocities in air and in sea water, although the acoustic properties of other substances require consideration when the transmitting and receiving devices are being dealt with.

For air at temperature t° C. the velocity $v = 1087 + 1{\cdot}81t$ ft. per second.

For sea water $v = 4756 + 13{\cdot}8t - 0{\cdot}12t^2$ ft. per second, according to the latest determination of Dr. A. B. Wood, for sea water having a salinity of 35 parts per thousand; the velocity being increased by about 3·7 ft. per second for each additional part per thousand in salinity. This gives a velocity of 1123 ft. per second in air, and 4984 ft. per second in normal sea water at a temperature of 20° C., so that the velocity in the sea is about four and a half times as great as in air.

Wave-length

As above mentioned, acoustic waves from a vibrating source consist of a number of compressions and rarefactions following one another and all travelling with a velocity given above. The distance between one compression or one rarefaction and the next is called the wave-length of the sound, and it is evident that if the frequency of vibration is n cycles per second there will be a train of n waves in a distance equal to the velocity, so that the wave-length $\lambda = \frac{v}{n}$. For example, if we take a frequency n of 500 $\sim$, the corresponding wave-length in air and in sea water respectively at 20° C. will be:

$$\text{In air } \frac{1123}{500} = 2{\cdot}246 \text{ ft., and in sea water } \frac{4984}{500} = 9{\cdot}968 \text{ ft.}$$

The greater wave-length in water introduces somewhat serious difficulties as regards directional transmission and reception, as will appear later.

Transmission of Sound through Various Substances

As was first shown by Newton, the velocity of sound in any substance can be calculated from a knowledge of its elasticity of volume and its density. If κ is the elasticity and ρ the density, it is easy to prove that the velocity of propagation of sound $v = \sqrt{\frac{\kappa}{\rho}}$. It must be remembered, however, that with the rapid vibrations of audible sounds the heating and cooling resulting from compression and rarefaction have no time to die away, and we must therefore take the adiabatic elasticity instead of the constant temperature or isothermal elasticity in the above formula. For gases the isothermal elasticity is equal to the pressure, or about 10^6 dynes per square centimetre in the case of ordinary atmospheric pressure, and the adiabatic elasticity of air is 1·41 times this amount, while the density of air ρ = 0·00129 gm. per cubic centimetre, so that the velocity $v = \sqrt{\frac{1{\cdot}41 \times 10^6}{0{\cdot}00129}}$ = 33,000 cm. per second, or about 1085 ft. per second, agreeing closely with the value obtained by direct experiment.

The mathematical theory also enables us to calculate the amount

of acoustic power transmitted by sinusoidal waves, and the phenomena which result when the sound passes from one medium into another—matters of considerable importance in connection with submarine signalling. It can be shown that the relation between the pressure P due to vibration (i.e. the alternating excess over the mean pressure), and the velocity V *of a moving particle*, at any point of the medium in the case of a plane wave of large area compared with the wave length, is given by the relation $P = RV$, where $R = \sqrt{\kappa\rho}$. This relation being analogous to Ohm's Law in Electricity, the quantity R has been called by Brillié the "acoustic resistance" * of the medium. The power transmitted (w) per unit area of the wave front is $\frac{1}{2}P_{max.}V_{max.}$, i.e. $P^2_{max.}/2R$, or $RV^2_{max.}/2$. For a plane wave sinusoidal disturbance of frequency n periods per second, and writing ω for $2\pi n$, we have $V_{max.} = \omega a$, where a is the amplitude of the displacement, so that $w = \omega a P_{max.}/2 = R\omega^2 a^2/2$ ergs per square centimetre per second.

For ordinary sea water in which $\kappa = 2{\cdot}2 \times 10^{10}$ dynes per square centimetre, and $\rho = 1{\cdot}028$, $R = \sqrt{\kappa\rho} = 14 \times 10^4$, so that for a frequency of 500 ~ and a displacement of 0·1 mm., the power transmitted would be 7 watts per square centimetre.

When sound passes from one medium into another, it can be shown that unless the two media have the same acoustic resistance there will be a certain amount of reflection at the interface. If r is the ratio of the acoustic resistance of the second medium to that in the first, and the wave fronts are parallel to the interface which is large in comparison with the wave length,

$$P_2 = \frac{2r}{r+1}P_1,\ P_1' = \frac{r-1}{r+1}P_1,\ V_2 = \frac{2}{r+1}V_1,\ \text{and}\ V_1' = \frac{r-1}{r+1}V_1,$$

where P_1 is the pressure and V_1 the velocity in the original wave,
P_2 " V_2 " transmitted "
P_1' " V_1' " reflected "

If the second medium is highly resistant compared with the first, so that r is very large, $P_2 = 2P_1$, $P_1' = P_1$, $V_2 = 0$, and $V_1' = V_1$, so that the pressure at the interface is double that in the original wave and the velocity, being equal to $(V_1 - V_1')$, is zero, since the movements in the direct and reflected waves are equal and in

* H. Brillié, *Le Génie Civil*, 23rd and 30th August, 1919; "Modern Marine Problems in War and Peace", 11th Kelvin lecture to Institution of Electrical Engineers, by Dr. C. V. Drysdale: *Jour. Inst. Elect. Eng.*, 58, No. 293, July, 1923, pp. 591–3.

opposite directions. The wave is therefore totally reflected back in the first medium and there is no transmission.

On the other hand, if the second medium has a very small acoustic resistance compared with the first, so that r is very small,

$$P_2 = 0, \quad P_1' = -P_1, \quad V_2 = 2V_1, \quad \text{and} \quad V_1' = -V_1.$$

In this case the total pressure $(P_1 + P_1')$ at the interface is zero, and the velocity $(V_1 - V_1')$ is $2V_1$, so that the velocity is doubled, but there is again no transmitted wave since $P_2 = 0$, and the wave is totally reflected with a reversal of the velocity V_1. In the first case the surface is called a " fixed " end, and in the second a " free " end.

If, finally, the two media have the same acoustic resistance, so that $r = 1$,

$$P_2 = P_1, \quad P_1' = 0, \quad V_2 = V_1, \quad \text{and} \quad V_1' = 0,$$

and the wave passes on without any reflection. This is the ideal condition to be secured in transmitters and receivers.

When the two acoustic resistances are not equal, it is easily shown that the ratio of the energy in the transmitted wave to that of the original wave, which we may call the efficiency of transmission (η), is $\dfrac{4r}{(r+1)^2}$, while that for the reflected wave is $\left(\dfrac{r-1}{r+1}\right)^2$. Now for water we have seen that $R_1 = 14 \times 10^4$, and for air $R_2 = 40$ (see also table on p. 292), so that $r = \dfrac{R_2}{R_1} = 2{\cdot}86 \times 10^{-4}$, and $\eta = \dfrac{4r}{(r+1)^2} = 0{\cdot}0011$, so that only a little over 0·1 per cent of the energy is transmitted. This at once illustrates the difficulty in all underwater listening, as the sound passing through the water must generally pass into the air before falling on the drum of the ear.

Again, the value of R for steel is about 395×10^4, so that on passing from water to steel $r = 28$ approximately, and the efficiency of transmission is about 13 per cent, while from steel to air it is only 0·004 per cent. Hence for sound to pass from water through the side of a ship to the air inside, the efficiency would be only 13 per cent of 0·004 per cent, or 0·00052 per cent, were it not for the fact that the plates of a ship are sufficiently thin to act as diaphragm, and thus allow a greater transmission than if they were very thick. In any case, however, the loss of energy is extremely great, and this has led to the practice of mounting inboard listening devices, either

directly on the sides of the ship or in tanks of water in contact with the hull, as will be described later.

The following table of the acoustic properties of various media has been given by Brillié.

Medium.	Value of κ (kg. per sq. mm.).	Value of ρ (C.G.S.).	Value of $\sqrt{\frac{\kappa}{\rho}}$ (velocity metres per second).	Values of $R = \sqrt{\kappa\rho}$ in C.G.S. units.
Steel	2×10^4	7·8	5100	395×10^4
Cast iron	$0{\cdot}95 \times 10^4$	7·0	3680	258×10^4
Brass	$0{\cdot}65 \times 10^4$	8·4	2780	234×10^4
Bronze	$0{\cdot}32 \times 10^4$	8·8	1910	168×10^4
Lead	$0{\cdot}06 \times 10^4$	11·4	725	$82{\cdot}5 \times 10^4$
Wood, Teak	$0{\cdot}16 \times 10^4$	0·86	4300	37×10^4
Wood, Fir	$0{\cdot}09 \times 10^4$	0·45	4470	20×10^4
Wood, Beech	$0{\cdot}06 \times 10^4$	0·8	2740	22×10^4
Water	2×10^2	1·0	1410	14×10^4
Rubber	Below 1 (variable according to the nature of the rubber)	1 (approximately)	Below 100	Below 1×10^4
Air	$1{\cdot}40 \times 10^{-2}$	0·0013	328	$0{\cdot}004 \times 10^4$

It should be noticed that the values of R for pine or beech wood are not greatly different from that for water, so that sound should pass from water to wood or vice versa without great reflection.

Pressure and Displacement Receivers

From what has been said concerning the theory of acoustic transmission, it is evident that sound may be detected either by the variations of pressure in the medium or by the displacements they produce, in the same way as the existence of an electrical supply may be detected by the electrical pressure or by the current it produces. Acoustic receivers may therefore be classed as pressure receivers, analogous to electrical voltmeters, and displacement receivers corresponding to ammeters; but this classification is not a rigidly scientific one, as a receiver cannot be operated by pressure or by displacement alone. We have seen that the power per unit area of wave front is $\frac{1}{2}P_{max.}V_{max.}$, so that unless the receiver makes use of both the pressure and velocity of displacement it receives no energy and can give no indication. A perfect pressure receiver would, in fact, constitute a fixed-end reflector, and a perfect displacement receiver a free-end reflector, in both of which cases we have seen no energy is transmitted.

The distinction between pressure and displacement receivers is, however, a useful one, just like that between a voltmeter and ammeter. A voltmeter is predominantly an electrical pressure-measuring device although it takes a small current, and an ammeter a current-measuring device although it requires a small P.D. across its coils. Similarly a pressure receiver is one in which the diaphragm is comparatively rigid and yields very little to the vibrations, while a displacement receiver is one with a very yielding diaphragm. The distinction is of importance directly we consider directional receivers, as the pressure in a uniform medium is the same in all directions while the displacements are in the line of propagation, so that a pressure receiver will give no difference of intensity on being rotated into different directions if it is so small that it does not distort the waves, whereas a displacement receiver will give a maximum when facing the source.

As regards sensitiveness, however, it is evident that the best results should be obtained when the receiver absorbs the whole of the energy which falls upon it, which will only be the case when a given alternating pressure on the diaphragm produces the same displacement as it does in the medium, so that the energy is completely transmitted into the receiving device without reflection. This will only be the case if the diaphragm is in resonance with the vibrations and the receiving mechanism absorbs so much energy as to give critical damping.

In the case of a properly designed receiver which is small in comparison with the wave length, it will draw off energy from a greater area of the wave front than its own area, just as a wireless aerial may absorb energy from a fairly large region around it.

The practical construction of underwater receivers and hydrophones will be dealt with later, but it will be well at this point to give some idea of their essential features. The simplest form of such a receiver, which is analogous to the simple trumpet for air reception, is what is called the Broca tube, which consists of a length of metal tube with a diaphragm over its lower end. When this is dipped into the water, the sound from the water is communicated through the diaphragm to the air inside the tube, and the observer listens at the free end. This is moderately effective, but not very sensitive or convenient, as it makes no provision for amplifying the sound, and it is not easy to listen through long bent tubes, so that the observer must generally listen only a few feet

above the water. Modern hydrophones are therefore nearly all of an electrical character, containing microphones or magnetophones from which electrical connections are taken to ordinary telephone receivers at the listening point.

Microphones are generally used, as they are more sensitive, and there are two types of microphone which correspond approximately to pressure or displacement receivers respectively. The former is termed the "solid back" type, in which a number of carbon granules are enclosed between a metal or carbon plate forming or attached to a diaphragm and a solid fixed block of carbon at the back. If pressure is applied to the diaphragm it compresses the granules and increases their conductivity, so that a greater current passes from a battery through the microphone and the receivers and reproduces the sound through the pressure variations. In the "button" type of microphone, on the other hand, the carbon granules are enclosed in a light metallic box or capsule covered by a small diaphragm, and the whole arrangement is mounted on a larger diaphragm, so that its vibrations move the capsule as a whole and shake up the granules, with only such changes of pressure as result from the inertia of the capsule. In this case it is the motion or displacement of the diaphragm which produces the variations of resistance in the microphone.

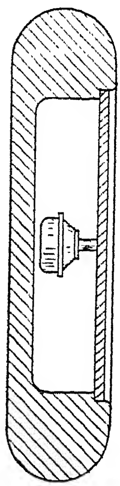

Fig. 1.—Non-directional Hydrophone

The commonest type of simple hydrophone is diagrammatically shown in fig. 1 and illustrated in fig. 18. It consists simply of a heavy circular metal case of disc form with a hollow space covered by a metal diaphragm to the centre of which a button microphone is attached. It is fairly sensitive but has no directional properties.

Directional Transmission and Reception

The problems of directional transmission and reception are among the most important as regards acoustic transmission. As in the case of wireless telegraphy or telephony, acoustic transmission suffers greatly from the difficulty that sound, like wireless waves, tends to radiate more or less uniformly in all directions, with the

result that its intensity rapidly diminishes according to the inverse square law, and there is great difficulty as regards interference and want of secrecy. Again, as regards reception, it is of comparatively little value to have a sensitive receiver which will detect the existence of a source of sound at a great distance if it gives no indication of the direction or position of the source. On this account the question of directional transmission and reception is of at least equal importance to that of powerful transmitters and sensitive receivers. This question of directional transmission and reception has received a large amount of attention.

The Binaural Method of Directional Listening. — Our own ears form a very efficient directional receiving system. When a sudden noise occurs we instinctively turn towards the source, and if we are blindfolded we can generally tell with considerable accuracy the direction from which a sound comes. This is due to the fact that, as our two ears are on opposite sides of the head and about 6 in. apart, the sound reaches one ear a little sooner than the other, unless it arises from a point in a plane perpendicular to the line joining the ears, i.e. directly in front of, behind, or above our head. Our ears are exceedingly sensitive to this minute difference of time, and as this interval depends upon the direction, getting larger the more the source is on either side, we learn to estimate the direction fairly closely, provided our two ears are nearly equally sensitive. This is known as the binaural (two-ear) method of estimating direction, and it has been developed both for air and submarine listening. For example, if we take two trumpets fixed to a horizontal bar (fig. 2), each of which is provided with a definite

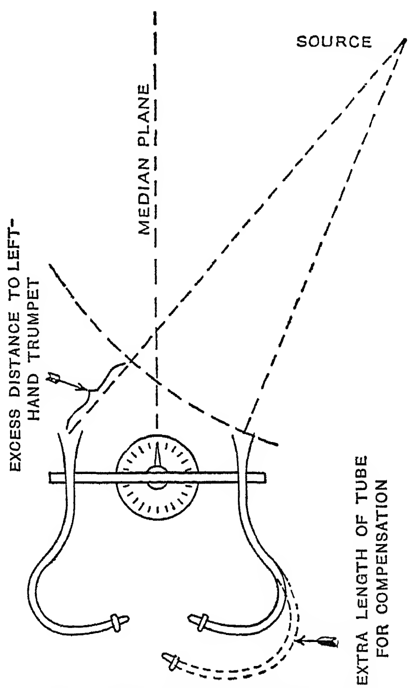

Fig. 2.—Binaural Listening with Trumpets

length of rubber tubing to an ear piece, we can detect and locate an aeroplane with considerable accuracy from the noise of its engines, as the trumpets magnify the sound, and the sensitiveness to direction may be increased by increasing the distance between the trumpets. When a sound is heard, the observer swings the bar carrying the trumpets round in the direction indicated, and as he does so the sound appears to cross over from one ear to the other behind his head. The position at which this occurs is called that of binaural balance, and when this balance is obtained the bar is at right angles to the direction of the source.

The same principle can obviously be applied to underwater listening with two receivers, but in this case it should be noted that, as the velocity of sound in water is about four and a half times that in air, the distance between the receivers must be increased in that proportion to obtain the same difference of time, and therefore equal binaural discrimination. As this involves the use of a somewhat long bar, which is troublesome to turn under water, recourse is generally had to what is called a binaural compensator for determining the direction.

Returning to our pair of trumpets in fig. 2, suppose that the source of sound is to the right of the median plane, and that the tubes from the trumpets, instead of being of equal length, are of different lengths, so that the additional length of tube to the right-hand trumpet is equal to the extra distance from the source to the left-hand trumpet. In this case it is evident that the delay of the sound in reaching the left-hand trumpet is balanced by the extra delay between the right-hand trumpet and the ear, and that binaural balance will be obtained although the source is on one side of the median plane. It is therefore possible to obtain the direction of a source with a fixed bar carrying the receivers, provided that arrangements can be made for varying the length of the stethoscope or ear tubes, and such an arrangement is called a binaural compensator, the most simple form of which is shown in fig. 3. Here the two equal tubes from the trumpets are brought to the two ends of a long straight tube, which is, however, made of three sections, the middle one sliding in the two end portions. The middle tube is blocked at its centre, and is provided with two apertures from which two equal rubber tubes are taken to the ear pieces. When the centre section of tube is in its middle position the two lengths of air path from the trumpets to the ear pieces are the same, and binaural balance will therefore only be obtained when the source is

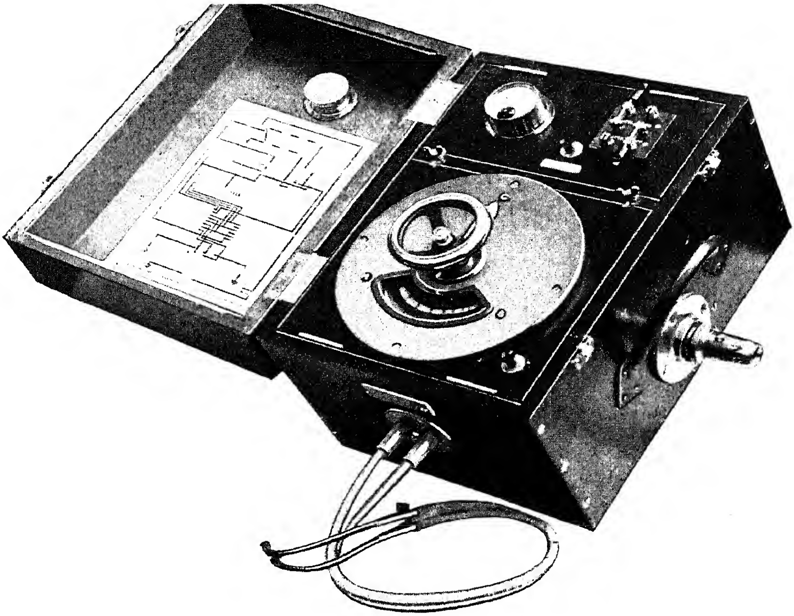

FIG. 4(*a*).—AMERICAN COMPENSATOR FOR DIRECTIONAL LISTENING

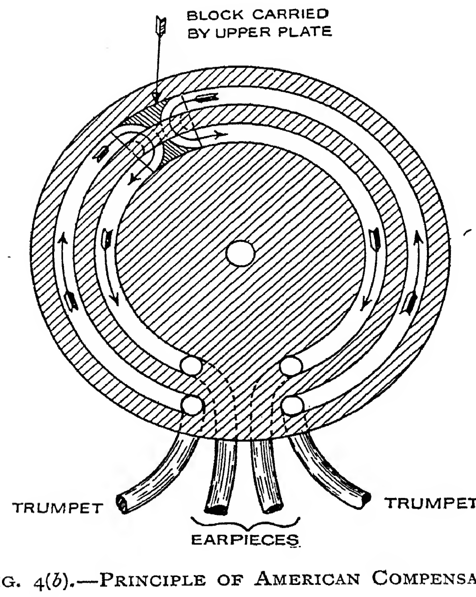

FIG. 4(*b*).—PRINCIPLE OF AMERICAN COMPENSATOR

in the median plane; but if the source is to the right of this plane, so that the sound reaches the right-hand trumpet first, sliding the centre tube to the left increases the path from the right-hand trumpet and diminishes that from the left-hand one, so that balance can be restored, and an index on the sliding tube will read off the direction on a suitably engraved scale which can be divided from the relation $2d = b \sin\theta$, or $\sin\theta = \frac{2d}{b}$, where b is the distance between the trumpets, d the displacement of the central tube from its mid position, and θ the angle of obliquity of the direction of the source from the median plane.

In order to carry out directional listening on these lines with the greatest convenience, a circular form of compensator has been designed in the United States and made by the Automatic Telephone Company. Fig. 4*a* shows the external appearance of this compensator, and fig. 4*b* the essential feature of its construction. Two concentric circular grooves are cut in a fixed plate, and are covered by a plate which converts them practically into circular tubes. This plate can be rotated above the fixed plate, and is provided with two projections which close the grooves but connect the inner and outer ones together by two cross channels. The sound from the two trumpets, entering the two ends of the outer groove, travels round this groove to the stop and then passes through the channels to the inner grooves, returning to its two ends, to which the ear pieces are connected. It is evident that as the upper plate is turned the difference of path between the two systems is altered by four times the distance through which the stop travels, and a pointer on the top plate indicates the direction on a dial.

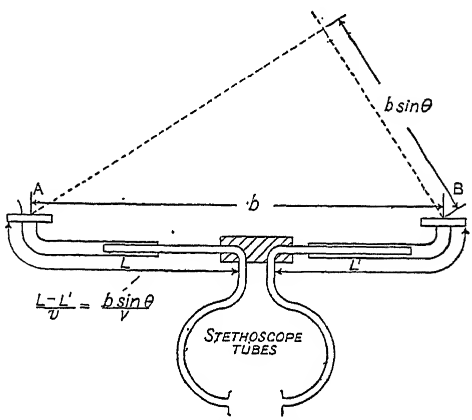

Fig. 3.—Binaural Method with Rectilineal Compensator

This binaural principle is of such importance that it has been described at length, and many applications of it will be seen later,

but there are other methods of directional reception which may first be referred to.

Sum-and-difference Method.—In the case of electrical receivers the binaural method may be replaced by what is called the sum-and-difference method. Suppose, in fig. 5, that our two trumpets on the bar are replaced by two similar ordinary microphone receivers M_1 and M_2, and that these receivers are connected to two telephone transformers T_1 and T_2, the secondaries of which can be

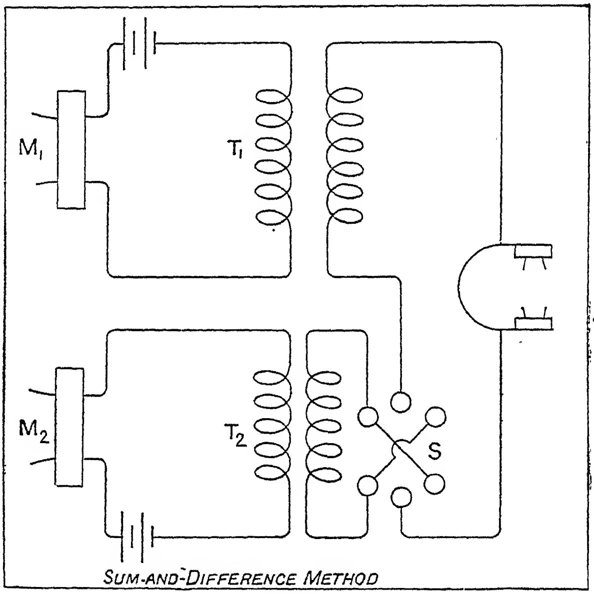

Fig. 5

connected in series as shown. It is evident that if the source is in the median plane, so that the sound reaches both receivers simultaneously, they should be similarly affected and produce equal electromotive forces in the transformer secondaries. If these secondaries are connected so as to assist one another, a loud sound should be heard, but if one of them is reversed by the switch S the two electromotive forces should be equal and opposite, and silence should result. But if this is the case, and the source moves to one or other side of the median plane, the sound will reach the two receivers at different times, and cancellation should no longer take place, so that the source should appear louder the greater the angle of reception from the median plane. By swinging the bar

until the sound vanishes, or at least becomes a minimum, the direction of the source is given just as in the binaural method, and in this position the sound will be a maximum when the transformers assist one another.

This sum-and-difference method has the advantage over the binaural method that it does not depend on the binaural sensitiveness of the observer, which may be very poor, especially in the case of persons with partial deafness in one ear; and on this account some observers prefer it. On the other hand, it reintroduces the objectionable feature of swinging the bar unless a compensator is introduced between two sets of receivers which introduces undesirable complication. But in any case this sum-and-difference method is of great value in connection with electrical receivers, as it brings out a difficulty which has to be overcome before such receivers can be used for binaural listening. It will be noticed that for silence to be obtained with the difference connection the sound must affect both receivers equally, but this is very rarely the case with ordinary microphones, owing to differences in the properties of their diaphragms. In fact, if two such receivers are placed close together so as to receive the same sound, it is not uncommon to find very little difference between the sound heard with the sum-and-difference connections, and in this case such receivers are quite useless for binaural listening, which depends upon perfect similarity of response. By replacing the ordinary metal or carbon diaphragms by rubber membranes, however, much greater equality can be secured, and the sum-and-difference method can be used in the test room to test this equality and to select perfectly paired receivers either for binaural or for sum-and-difference direction finding.

Directional Receivers.—It has already been pointed out that although the pressure changes in an acoustic beam have no direction, the displacements take place in the direction of propagation, and that a displacement receiver should therefore have directional properties. This principle has not actually been employed for directional listening to any extent, but Mr. B. S. Smith has devised a displacement receiver consisting of a small hollow sphere containing a magnetophone transmitter, the whole arrangement being of neutral buoyancy. Such a sphere vibrates as if it were part of the water, and consequently gives maximum effect on the magnetophone when its axis is in the direction of propagation and zero when it is perpendicular to it.

The type of directional receiver which has been most employed

in practice, however, is of a balanced type, as shown in figs. 6 and 7. It is similar to the non-directional hydrophone (fig. 1), except that instead of a thick hollow metal

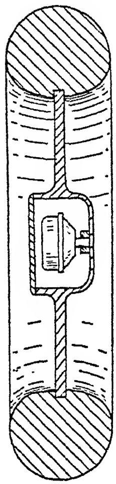

Fig. 6.—Bi-directional Hydrophone

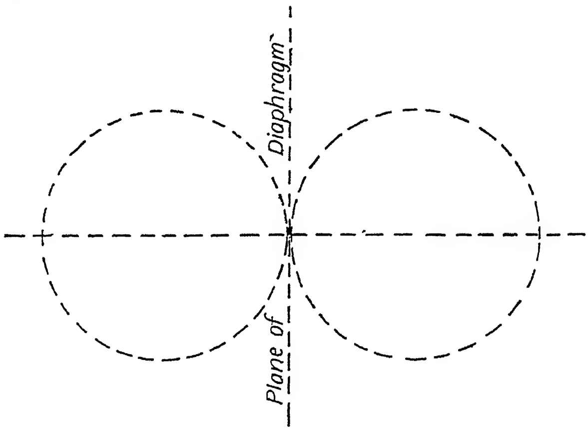

Fig. 7.—Polar Curves of Intensity for Bi-directional Hydrophone

case it has simply a heavy brass ring with a central diaphragm having a hollow boss at its centre in which the button microphone is fixed. Obviously if such an arrangement is placed so that its plane lies along the direction of propagation, the pressure falls upon both faces of the diaphragm equally and simultaneously and no motion results, so that nothing can be heard in this position. When the hydrophone is turned with one of its faces towards the source, however, the back face is screened by the ring, and the sound reaches it later and with less intensity, so that there is a resultant effect. On turning such a hydrophone round, therefore, the sound is a minimum when the edge points towards the source, and rises to a maximum when turned through a right angle, the intensity for various angles of turning being shown in the polar diagram fig. 7. A similar effect is given by the Morris-Sykes directional hydrophone (fig. 8), which has two similar diaphragms on its two faces connected by a rod at their centres, on which the microphone is mounted. As the variations in pressure tend to move the two

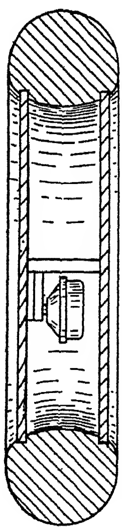

Fig. 8.—Morris-Sykes Hydrophone

diaphragms in opposite directions, no movement of the bar is produced and no sound heard when the hydrophone is edge on.

These forms of directional hydrophone are fairly effective, giving a fairly sharp minimum, but they do not entirely fill the requirements for directionality, as it is evident that minimum is given when either edge of the disc points to the source, so that the source may be in either of two diametrically opposite directions. For this reason they are called bi-directional hydrophones; but it has been found possible to get over this difficulty and to convert a bi-directional into a uni-

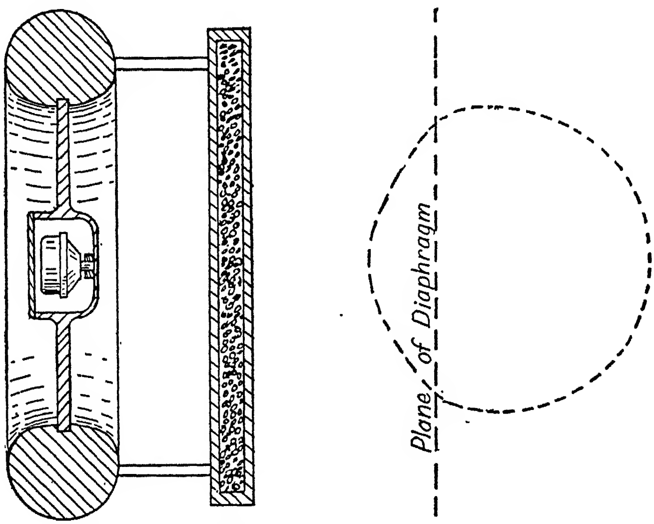

Fig. 9.—Uni-directional Hydrophone and Polar Curve of Intensity

directional hydrophone, by simply mounting what is called a "baffle plate" a few inches away from one face, as shown in fig. 9. This baffle plate may be made of layers of wood or metal or have a cavity filled with shot in it, so that it tends to shield the sound from one face. Such a hydrophone gives the loudest sound when the unbaffled face is turned towards the source and the weakest sound when it is turned directly away from it, the intensity in various directions being shown by the polar curve, so that there is now no ambiguity as to direction, and the device is then called a uni-directional hydrophone. It does not, however, give such definite indications of direction as the sharp minima of the bi-directional form, and it is therefore better to couple a uni-directional and a bi-directional hydrophone at right angles to one another on

the same vertical shaft. When maximum intensity is observed on the former and a minimum on the latter, the direction of the source is definitely given.

Besides the foregoing methods of directional reception there are others, such as those of Professors Mason and Pierce, depending on the principle of acoustic integration first enunciated by Professor A. W. Porter, which leads to the use of large flat surfaces for reception, and the Walser gear in which the sound is brought to a focus by a lenticular device, as will be described below.

As regards directional transmission, it may first be mentioned as a general principle of all radiation that transmission and reception are reciprocal problems, that good receivers make good transmitters, and that a directional receiver will make a directional transmitter with the same distribution of intensity in different directions. For example, if, instead of listening by means of two trumpets coupled by equal tubes to the ear, we bring the two tubes to a powerful source of sound so that the sound escapes in an exactly similar manner from the two trumpets, an observer in median plane will hear this sound very loudly, but as he moves to one or other side of this plane the sound will appear fainter. Similarly, by vibrating the diaphragm of a uni-directional hydrophone sound will be emitted chiefly in one direction, and by extension of this principle a beam of sound may be sent in any direction we please.

PRACTICAL UNDERWATER TRANSMITTERS AND RECEIVERS

We can now turn to the actual devices employed for submarine signalling, and they may be described under the headings (*a*) transmitters, (*b*) receivers, and (*c*) directional devices.

Submarine Transmitters or Sources of Sound

The simplest form of submarine transmitter is the submarine bell which has been used as an aid to navigation for many years. Originally suggested by Mr. Henry Edmunds in 1878, it was not until 1898 that it was taken up seriously as a practical navigational device by Mr. A. J. Munday and Professor Elisha Gray, who formed the Gray Telephone Company in 1899, and employed a bell struck under water with a submerged telephone receiver. After Professor Gray's death in 1901, the work was carried on by Mr.

Munday, who started the Submarine Signal Company to take over the operations. Various forms of submarine bell were experimented with, but the form which was finally adopted is shown in fig. 10, and consists of a bronze bell, weighing 220 lb. and having a frequency of 1215 ~ in water, which is struck by a hammer generally operated by compressed air. A twin hose pipe is used to supply the compressed air and to convey away the exhaust air from the apparatus, and the strokes are regulated by a "code valve", which consists of a small diaphragm actuating the main air supply to the hammer mechanism. This type of bell is generally used on light-ships, in which case it is simply hung overboard to a depth of 18 to 20 ft., but in the case of lighthouses where electric supply is available an electrically operated bell of the same type is employed, which is hung on a tripod stand about 25 ft. high and 21 ft. spread, standing on the bottom in any convenient position up to a mile or so from the lighthouse. In this case the hammer is operated by a circular iron armature attracted to six electromagnets on a common yoke, the pole faces being covered by copper cups to prevent sticking by residual magnetization. A four-core cable is provided, two for supplying the $3\frac{1}{2}$ amp. of operating current, the other two being connected to a telephone transmitter in the mechanism case, which enables the operator to hear if the bell is working properly. The first of these electrically operated bells was laid down at Egg Rock, near Boston Harbour, United States, and a large number of pneumatically and electrically actuated bells are now in service round the British and American coasts.

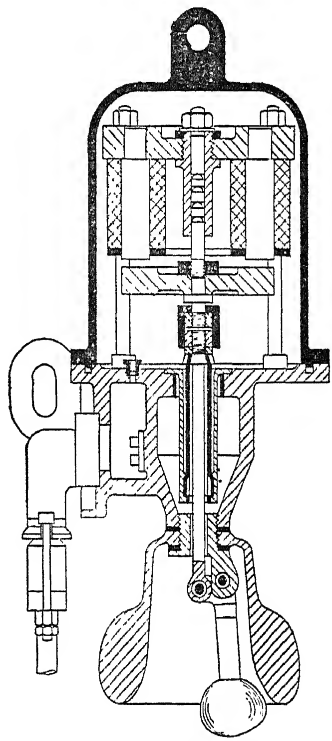

Fig. 10.—Submarine Signal Company's Bell. Electrically operated type.

Electromagnetic Transmitters

On account of the ease of the operation and control, electromagnetic transmitters have been most popular, and they are now made up to large sizes transmitting hundreds of watts of acoustic power. They may be divided into two classes: (*a*) continuous, and (*b*) intermittent or impulse transmitters.

(*a*) **Continuous Electromagnetic Transmitters.**—In all these transmitters alternating current is employed, of frequency corresponding to the natural vibration frequency of the vibrating system, and this current may be used either to energize a laminated electromagnet which acts on the diaphragm, or to traverse a coil in a powerful steady magnetic field, thus developing an alternating force which can be applied to the diaphragm. These two types of transmitter may be called the "soft-iron" and the "moving-coil" types respectively. In the former type the frequency of the note is double that of the alternating current as the diaphragm is attracted equally when the current flows in either direction; but in the latter type the note frequency is the same as that of the current.

The soft-iron type of continuous transmitter has been greatly developed by the Germans, and fig. 11 shows one of the most generally used types constructed by the Signalgesellschaft of Kiel. The diaphragm D is provided with a boss at its centre, to which is fixed a casting carrying a laminated E-shaped iron core C nearly in contact with a similar block of stampings C′ above. The exciting coil encircles the inner pole of these stampings, as in the familiar core type of transformer, and produces a powerful attractive force at each passage of the current in either direction, so that the frequency of variation of the force is double that of the current. The upper block of stampings is not rigidly fixed, but is coupled to the lower block through the agency of four vertical steel tubes T with steel rods inside them, the lengths of these rods and tubes being such that the natural frequency of their longitudinal vibrations is equal to that of the diaphragm. The diaphragm is bolted to a conical housing with glands for the introduction of the supply cables. A transmitter of this type, having a total weight of about 5 cwt. and a diaphragm about 18 in. diameter, gives an acoustic radiation of 300 to 400 watts, the mechanical efficiency being about 50 per cent.

A great objection to these moving iron transmitters is their inherently low power factor owing to their great inductance, which involves a large wattless exciting current. This can, of course, be

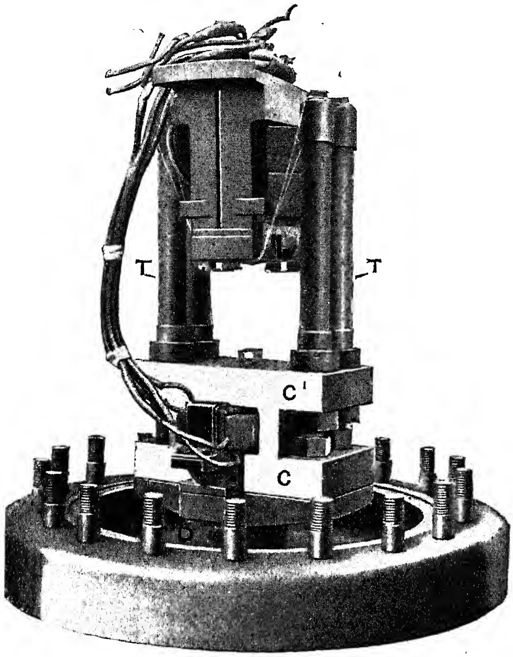

FIG. 11.—CONTINUOUS ELECTROMAGNETIC TRANSMITTER AS CONSTRUCTED BY THE SIGNALGESELLSCHAFT OF KIEL.

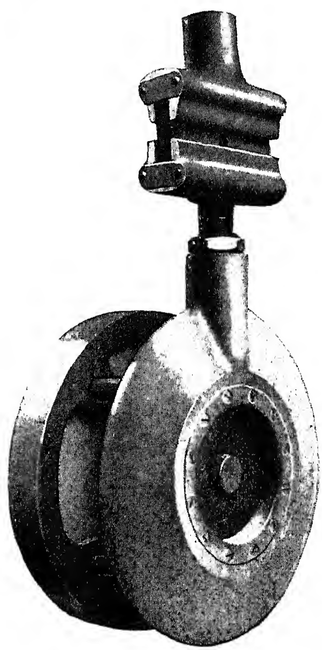

FIG. 20—SINGLE-DIAPHRAGM BI-DIRECTIONAL HYDROPHONE CONVERTED INTO UNI-DIRECTIONAL INSTRUMENT BY ADDITION OF BAFFLE PLATE.

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rectified by using a large condenser in parallel or series with the exciting coils, but this is not a very satisfactory expedient.

On this account the moving-coil type of transmitter has been favoured, especially by the Americans, and its fundamental principle is diagrammatically shown in fig. 12. The coil of wire traversed by the alternating current is attached directly to the diaphragm, and moves in the annular field of a powerful "pot magnet" excited by direct current. This type has relatively little inductance, and therefore a high power-factor, but its construction is mechanically difficult, as the coils of wire do not form a rigid mass and are therefore liable to cause great damping and loss of efficiency.

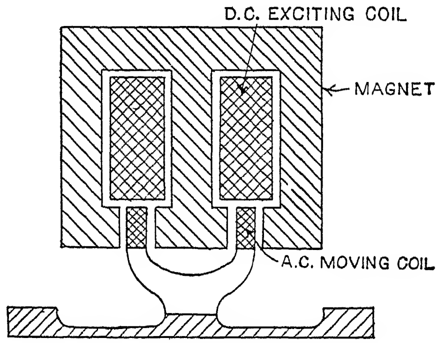

Fig. 12.—Principle of Moving-coil Transmitter

This difficulty was very neatly got over by Fessenden in the United States, and the Fessenden transmitter is probably the most efficient and powerful of all electromagnetic transmitters. The principle is exactly the same as above, but, instead of mounting the coil directly on the diaphragm so as to move with it, Fessenden employs a fixed coil which induces currents in a copper cylinder by transformer action, and this copper cylinder is attached to the diaphragm. Fig. 13 shows a diagrammatic section of a Fessenden transmitter in which the direct-current electromagnet is bipolar and encircles the copper cylinder which is attached to the diaphragm. The alternating current traverses a fixed coil wound on an inner iron core, the coil being wound in two halves in opposite directions to correspond with the two poles of the magnet, and this coil

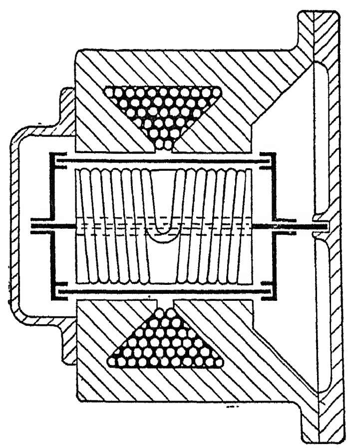
Fig. 13.—Diagram of Fessenden Transmitter

induces powerful currents in the copper cylinder which traverse the strong field of the magnet and impart longitudinal forces to it of the same frequency as that of the alternating current. The arrangement is therefore very rigid mechanically, and a high power-factor and efficiency are obtained at resonance, which is usually for a frequency of 500 ~. Transmitters of this type giving an acoustic radiation of 500 watts or more have been constructed, and are capable of signalling under water to a distance of 300 miles or thereabout. Morse signals can be sent by either of the above types of transmitter by the aid of a suitable signalling key.

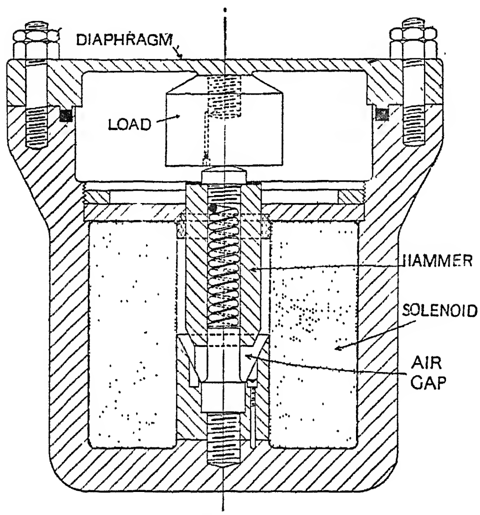

Fig. 14.—Diaphragm Sounder

(*b*) **Intermittent or Impulse Transmitters.**—Reference has already been made to the submarine bell, which was the first type of intermittent submarine transmitter and which can be operated electromagnetically. A more simple type of impulse transmitter is the diaphragm sounder of Mr. B. S. Smith, which has the advantage over the bell in that the striking mechanism is totally enclosed and therefore does not work in water. Fig. 14 shows a section of a sounder of this type, which is provided with an ordinary steel diaphragm with centre boss against which a cylindrical hammer strikes. This hammer is withdrawn on passing direct current through the exciting coil, against the force of a spiral spring, and upon the sudden interruption of the current the spring causes the hammer to strike the diaphragm with a single sharp blow, thereupon rebounding and leaving the diaphragm free to vibrate. A very powerful impulse, though of brief duration, owing to the heavy damping of the water, is produced in this way.

Similar powerful impulse transmitters have been constructed in which the hammer is operated pneumatically by compressed air at a frequency of about a hundred blows per second, and this type of

transmitter can be used for signalling in the Morse code, by means of a suitable pneumatic key.

A simple transmitter has been specially designed by the author for acoustic depth sounding, the object being to give a series of single impulses to the water without vibration. Here the elastic diaphragm is entirely done away with and its place taken by a square laminated plate, which is attracted to an E-formed laminated magnet on passing direct current round an exciting coil encircling the centre pole. The attractive force is $\frac{B^2}{8\pi}$ dynes per square centimetre, where B is the magnetic field in gausses, so that if $B = 15{,}000$, the force is about 14 Kgm. per square centimetre, and a pole area of 140 sq. cm. gives a total force of about 2 tons. In order to impart this force to the water, the pole faces and plate are grooved, and india-rubber strips inserted which are compressed by the attraction of the magnet. On switching on this transmitter to a 100-volt circuit the current rises comparatively slowly, owing to its great inductance, and the plate is gradually drawn up, but on suddenly breaking the current the reaction of the rubber strips shoots the plate suddenly forward with an initial force of about 2 tons, and imparts a single sudden shock like an explosion to the water. The use of this transmitter will be explained in connection with acoustic depth sounding.

Submarine Sirens

A number of forms of submarine siren, in which plates or cylinders provided with holes through which jets of water pass when the plates or cylinders are rotated, have been devised both in this country and in Germany, and are extremely powerful. By suitably bevelling the holes, the water pressure can, of course, be made to rotate the plates, but this is objectionable from the signalling point of view, as it involves a gradual running up to speed and a consequent variation in the frequency of the note. On this account the plate or cylinder is usually rotated independently at a constant speed by an electric motor, and signalling is effected by switching on and off the high-pressure water supply. These sirens have not, however, come greatly into use, as the electromagnetic transmitters are so much more convenient, and they will therefore not be described in detail.

There are many other forms of acoustic transmitters, but the

above are most generally useful for acoustic signalling or impulse transmission. For sound-ranging purposes small explosive charges are sometimes employed.

Receivers or Hydrophones

The C Tube

The earliest and most simple of all subaqueous acoustic receivers as already mentioned was the Broca tube, consisting of a length of metal tube with a diaphragm stretched over its lower end. The Americans have improved this form of tube, by replacing the diaphragm by a thick-walled rubber bulb or teat, and have called it the C tube (fig. 15) from Dr. Coolidge, its inventor. It is fairly sensitive, but the amount of energy communicated to the air within the bulb is very small by the principle of transmission given above, and it suffers from the inconvenience of requiring the observer to listen at the end of a somewhat short tube.

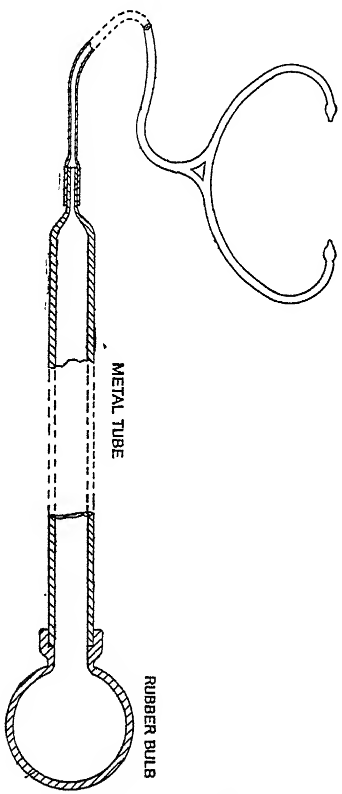

Fig. 15.—C Tube

The advantages, as regards sensitiveness and convenience, of employing microphones were also appreciated by the Americans who enclosed microphones in hollow rubber bodies, and a combination of three such bodies was often floated on a triangular frame and employed for binaural listening. Fig. 16 shows a double C tube arrangement for binaural listening. As has already been explained, binaural listening on two receivers permits the direction of the source to be ascertained, but it is evident that the direction suffers from the same ambiguity as in the bi-directional hydrophone, as a source symmetrically situated on the other side of the line joining

the two hydrophones would give the same difference of time of arrival. By using three hydrophones arranged at the corners of an equilateral triangle, and binauralling on each pair in turn, this ambiguity disappears. The necessity for correctly pairing the microphones by the sum-and-difference method has been already referred to.

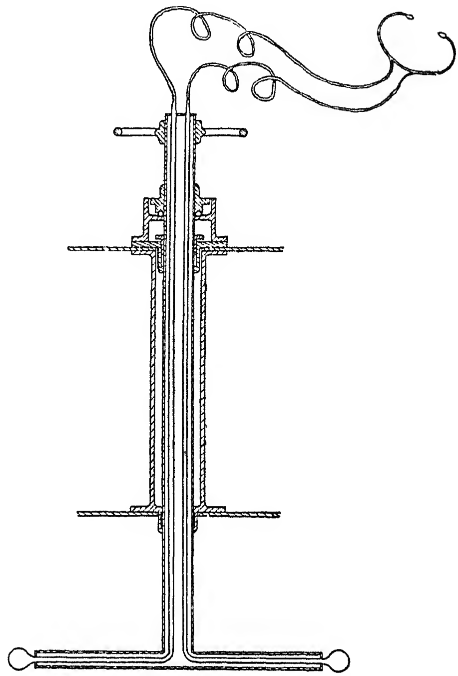

Fig. 16.—Double C Tube Binaural Arrangement

Magnetophones

Although greatly inferior in sensitiveness to microphones, magnetophones have some advantages for underwater listening, as they are free from the vagaries of granular microphones and can be more easily paired for binauralling. As their sensitiveness can be enhanced to almost any extent by the modern valve amplifiers, which cannot be employed with microphones owing to the grating or "frying" noise produced by the granules, they can be made equally effective.

The Fessenden transmitter described on p. 305 can be used as a powerful magnetophone receiver by exciting its magnet and listening on the coils, which are supplied with alternating current when transmitting, and it is commonly used as a receiver in signalling, as it is, of course, in tune with the note of all such transmitters. This sharp tuning, however, renders it unsuitable for general listening purposes.

One of the most effective magnetophone devices for inboard listening is the "air-drive" magnetophone of Mr. B. S. Smith (fig. 17). It consists of a massive lead casing (4) fixed to the side of the ship, carrying a thick india-rubber diaphragm (2) in contact with the water. Close behind this diaphragm an ordinary Brown reed-type telephone receiver (3) is mounted, so that the sound transmitted

from the water to the air behind it causes the diaphragm and reed of the receiver to vibrate and induces currents in the receiver windings. This type of receiver connected to a three-valve amplifier and high-resistance telephones gives a fairly faithful reproduction of ordinary sounds; and if four of these receivers are mounted on the hull in positions fore and aft and port and starboard, the screening effect of the hull enables the direction of the source to be estimated from the relative intensities on the four receivers—a four-way change-over switch being interposed between the receivers and the amplifier. Ship noises are greatly diminished by fixing the lead

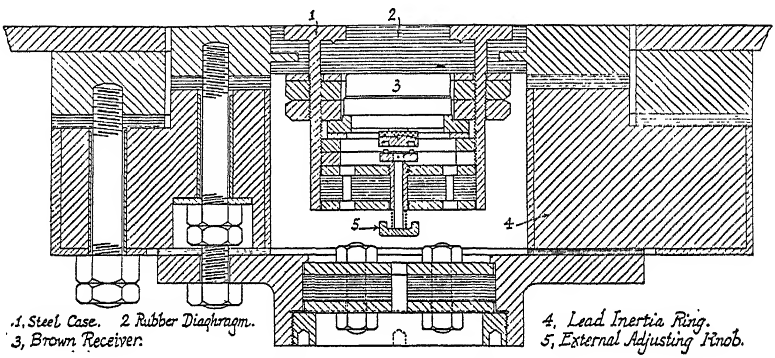

Fig. 17.—Air-drive Magnetophone

ring to the plates with a rubber seating, as the great inertia of the lead (4) prevents it from taking up the hull vibrations readily.

There are many other forms of receivers, but the above are the principal ones which have been used for underwater acoustic reception.

Practical Construction of Hydrophones

A few illustrations may now be given of the actual forms of some of the most generally used hydrophones. Fig. 18 shows the simplest form of non-directional hydrophone, of which a diagram was given in fig. 1, in which a heavy hollow bronze casting is provided with a diaphragm on one side, to the centre of which a small "solid back" microphone is attached.

Fig. 19 is an illustration of the double-diaphragm bi-directional hydrophone, diagrammatically shown in fig. 8, and fig. 20 (see plate facing p. 316) shows a single-diaphragm bi-directional hydro-

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FIG. 18.—NON-DIRECTIONAL HYDROPHONE

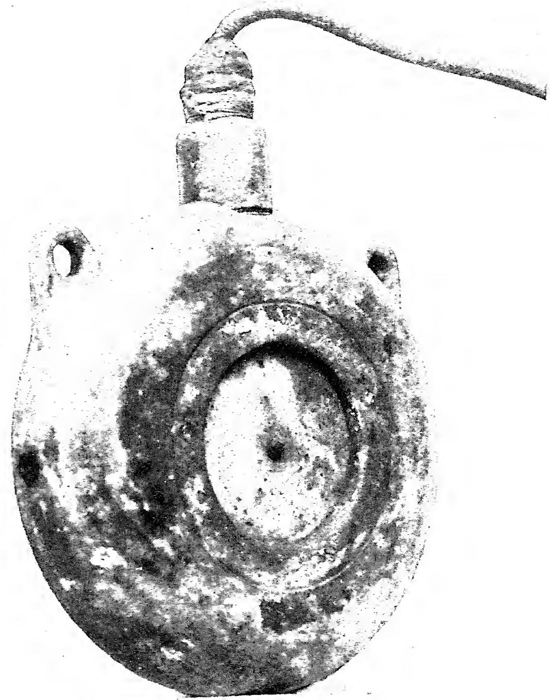

FIG. 19.—DOUBLE-DIAPHRAGM BI-DIRECTIONAL HYDROPHONE

phone converted into a uni-directional instrument by the addition of a baffle plate, as in fig. 9.

In order to be able to listen from a ship in motion and to reduce ship and water noises as much as possible, hydrophones, either of the rubber-block form or of one of the foregoing types, have been enclosed in fish-shaped bodies and towed through the water some distance astern, and combinations of such bodies have been used for directional listening by binauralling. The modern tendency,

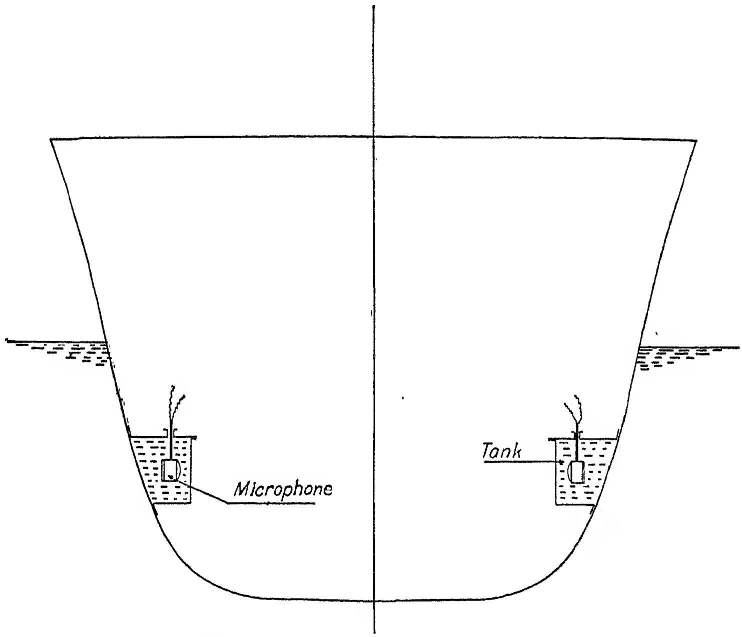

Fig. 21.—Reception by Hydrophone in Tanks

however, has been in the direction of inboard listening, by securing efficient acoustic insulation from the hull.

The method of listening in tanks inside the hull, first introduced by the Submarine Signal Company, has been greatly adopted by the Germans. Fig. 21 shows the disposition of a pair of these tanks with the hydrophones inside. This device avoids the great loss by reflection on passing from water to air, as has been referred to above.

A remarkably interesting and effective form of directional inboard listening device, however, is that known as the Walser gear, devised by Lieutenant Walser of the French navy, in which the

sound is brought to a focus, as in a camera obscura, and the direction determined by the position of this focus. For this purpose a "blister", consisting of a steel dome A of spherical curvature and about 3 ft. 6 in. diameter, part of which is seen in fig. 22, is fitted to the hull, and this steel dome is provided with a large number of apertures B into which thin steel diaphragms C are inserted. These

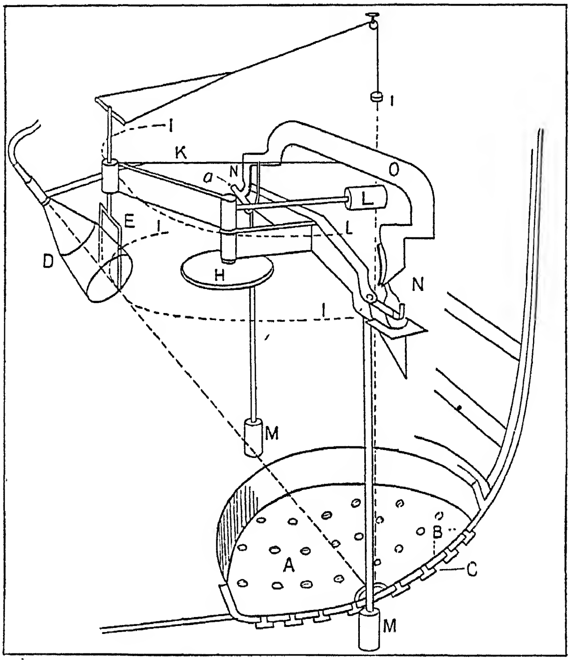

Fig. 22.—Walser Apparatus

diaphragms being on the spherical dome collect the sound and direct it to a focus at a distance of 5 or 6 ft. A trumpet D, to which a stethoscope tube is attached, is mounted on an arm E turning on a vertical axis, so as to be able to follow the focus and point in the direction of the sound from whatever direction it comes. Two of these blisters are generally mounted somewhat forward on the two sides of the hull, and an observer seated between them applies the tubes from the two trumpets to his ears, so that he can follow the position of the source on either side, the direction being given on a scale when the maximum intensity is obtained.

Directional Devices

Sound Ranging

One of the most important acoustic applications in the War was that of sound ranging for the detection of the position both of guns and of submarine explosions, the importance of which is obvious. There are two chief methods of location, which may be described as "multiple-station" and "wireless-acoustic" sound ranging respectively, but the former, although less convenient, was the only one employed in the War, as it needs no co-operation on the part of the sending station.

Multiple-station Ranging.—The multiple-station method

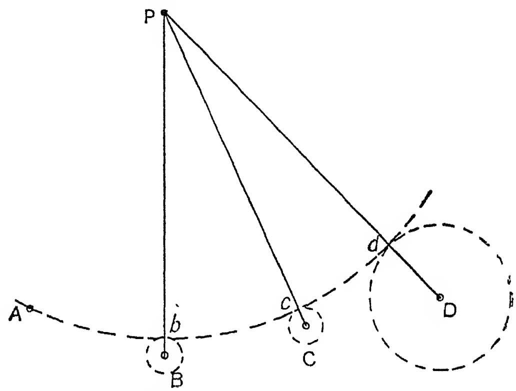

Fig. 23.—Sound-ranging Diagram

of sound ranging depends on the principle that sound waves are sent out as spheres with centre at the source of sound. If three or more receivers are therefore set up on a circle with centre at the source, the sound will arrive at all of them simultaneously, so that if the signals are all coincident the source must be at the centre of the circle passing through the receivers. If, however, the source is in any other position the signals will be received at different times, and if the differences of the times of reception are measured the position of the source can be located by calculation, or graphically.

A simple diagram (fig. 23) will make this method clear. Let ABCD be four receivers in any accurately known positions and P be the position of an explosion to be located. If we draw a circle with P as centre through the receiver A, it is evident that when the sound arrives at A it still has the distances *b*B to travel before arriving

at B, and cC and dD before arriving at C and D respectively, so that the times of arrival at B, C, and D are $t_1 = \frac{b\mathrm{B}}{v}$, $t_2 = \frac{c\mathrm{C}}{v}$, and $t_3 = \frac{d\mathrm{D}}{v}$ behind that at A. Consequently if we can measure the time intervals t_1, t_2, and t_3, and multiply them by the velocity, we get the perpendicular distances of the station B, C, and D from the circle passing through the source, and if we draw circles round B, C, and D with radii to scale representing these distances, the centre of a circle tangential to these circles will be the position of the source P.

The method of determining these time differences almost entirely employed during the War was by means of a multiple-stringed Einthoven galvanometer, four of these strings being connected to four microphones or hydrophones, while a fifth was connected to an electric clock or tuning fork, so as to give an accurate time scale. The image of the strings was focused on a continuous band of bromide paper, which was drawn through the camera and a developing and fixing bath by means of a motor, so that it emerged from the apparatus ready for washing and drying, though the times could be read off instantly it appeared. To facilitate the reading off of the time intervals, a wheel with thick and thin spokes was kept revolving in front of the source of light by means of a "phonic motor" in sychronism with a tuning fork, so that a number of lines were marked across the paper at intervals of hundredths and tenths of a second.

Fig. 24 is a reproduction of a sound-ranging record so obtained, on which the times of reception at four receivers are marked, and fig. 25 a view of the Einthoven camera outfit employed. The receivers used in this case were simple microphones, mounted on diaphragms bolted on watertight cases mounted on tripods lowered on the sea bottom and accurately surveyed, the microphones being connected by cables to the observing station.

On account of the importance of sound ranging as a means of locating the position of a ship in a fog, efforts have been made to improve it still further, and to eliminate the photographic apparatus. The greatest achievements in this direction have been made by Dr. A. B. Wood and Mr. J. M. Ford at the Admiralty Experimental Station, who have devised what they call a "phonic chronometer" for indicating the time intervals directly on dials to an accuracy within one-thousandth of a second. The principle of the instrument is very simple, and can readily be understood by reference to fig. 26. A phonic motor with vertical spindle revolves with a

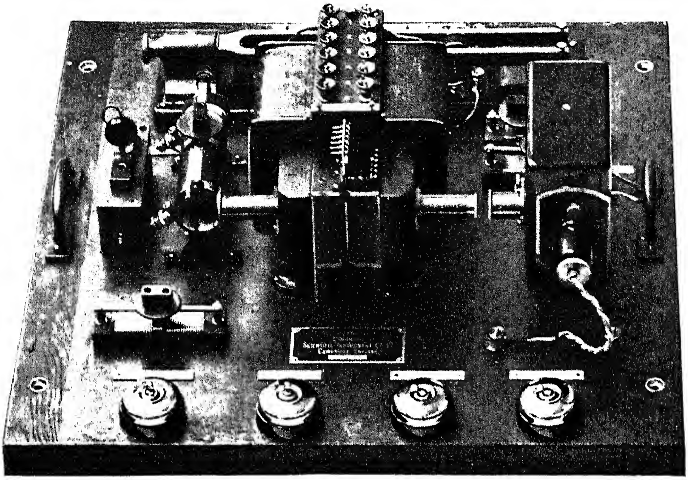

FIG. 25—EINTHOVEN CAMERA FOR SOUND RANGING

FIG. 26—THREE-DIAL PHONIC CHRONOMETER

Facing page 326

constant speed of 150 r.p.m. from a 25 ~ tuning fork, and at the top of this spindle is a circular vessel containing mercury to assist in maintaining constancy of speed, with a rim forming a steadily revolving disc. Around this disc can be mounted any number of recording mechanisms (in this case three), each of which consists of a light aluminium wheel mounted on a vertical spindle and the rim of which is only a few thousandths of an inch from the rim of the revolving disc. An electromagnet is provided with two similar

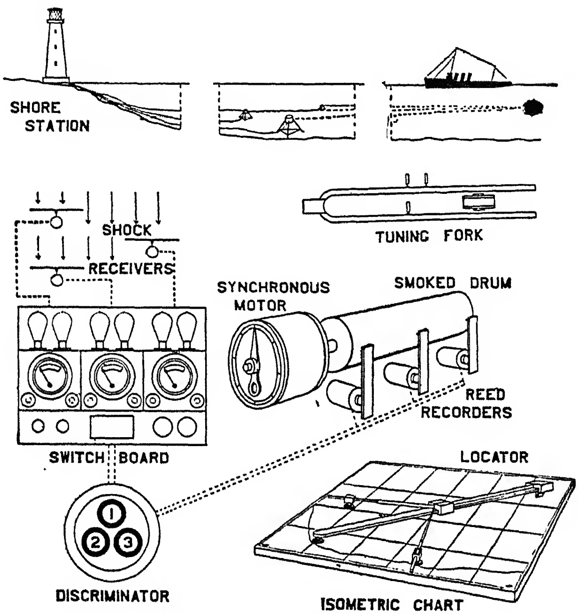

Fig. 27.—Sound-ranging Apparatus

windings in opposition, so that if neither or both of the windings be excited it is unmagnetized, but if a current passes through either winding separately it causes the magnet to move the spindle and bring the small wheel in contact with the revolving one, which causes it to revolve at the rate of 10 r.p.sec. A light aluminium pointer attached to this spindle revolves over a dial having a hundred divisions, so that each division represents one-thousandth of a second, and two smaller pointers geared to the spindle indicate time intervals up to 10 sec.

When employed for sound ranging the circuits of this chronometer are connected, as in fig. 27, to three or four hydrophones, which con-

sist in this case of diaphragms with single-point contacts which are thrown off on arrival of the shock, and remain broken until they are restored by electromagnets. Each of these contacts is connected to the electromagnet windings of the dials as shown, and it will be seen that as each contact is broken it breaks one of the circuits in either one or two of the dial mechanisms, and starts the pointers revolving until the breaking of another contact breaks the second winding and allows the small wheel to fly away from the revolving wheel and against a brake which immediately stops it. After the shock is received at all four hydrophones, therefore, the three dials indicate the time intervals between the arrival at the first hydrophone and that at the other three directly in thousandths of a second, each thousandth representing a distance of about 5 ft., from which the graphical diagram shown in fig. 23 can be constructed and the position of the source indicated on a chart.

In order to obtain this position as readily as possible the writer has devised what he calls a sound-ranging locator (fig. 28, see plate facing p. 326). It consists of a long steel bar pivoted at one end on a ball-bearing, the centre of which can be fixed on the chart exactly over the position of one of the hydrophones. Three thin steel bands are attached to the other end of this bar by means of keys, like the strings of a violin, and pass through a slot in a sliding piece to graduated rods sliding through similar ball-bearing swivels, which are fixed on the chart in positions corresponding to those of the other three hydrophones. The graduations on the sliding bars are marked in times to the scale of the chart, so that by sliding them to the readings corresponding to the time differences indicated on the chronometer, each strip is lengthened by the amounts *b*B, *c*C, and *d*D in the diagram fig. 23, and when the slotted slider on the main bar is pushed down and the bar turned until all the strips are tight, the point from which they radiate indicates the position of the source on the chart without any calculation, and a marking point just under the edge of the slider can be depressed to prick the position. In order to secure accuracy, each of the strips is provided with a small tension indicator which shows when the strip is strained to a definite tension.

Two strips only are shown in fig. 28, but any number may be employed according to the number of receivers.

A device on a similar principle has been put forward by Mr. H. Dadourian in the United States.*

* *Physical Review*, August, 1919.

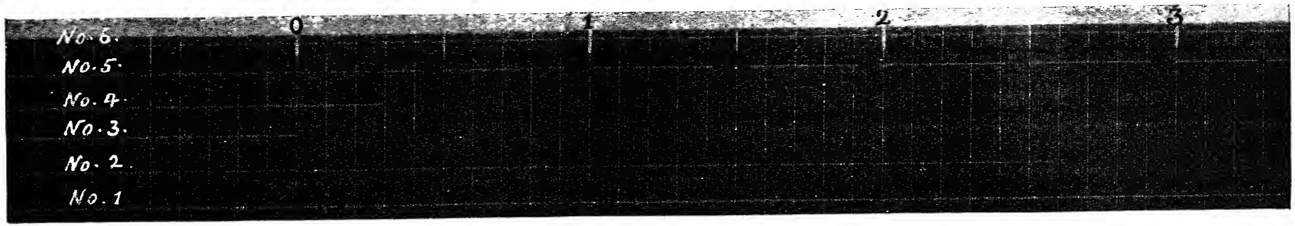

FIG. 24.—SOUND-RANGING RECORD

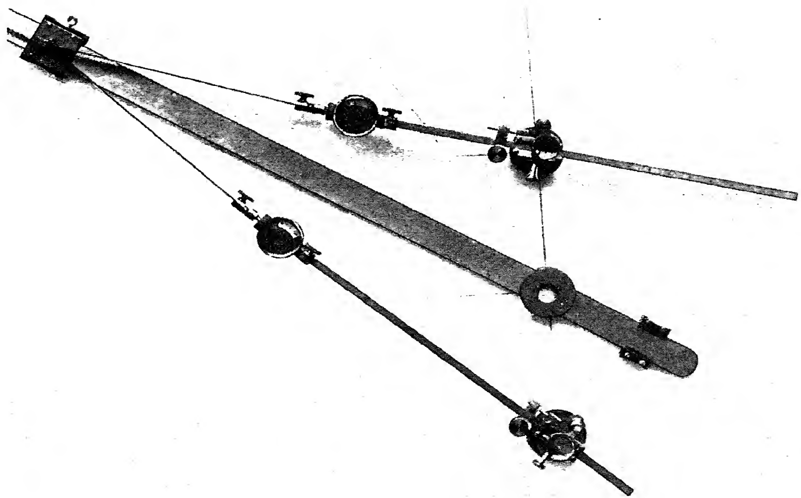

FIG. 28.—SOUND-RANGING LOCATOR

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In using the multiple-station method of sound ranging for assisting navigation in foggy weather, a ship desirous of being informed of its position calls up the nearest sound-ranging station, which instructs it to drop a depth charge. As soon as the record is received on the Einthoven camera or phonic chronometer, the position of the ship is worked out or marked by the locator and wirelessed to the ship.

Wireless-Acoustic Sound Ranging.—A method of sound ranging which promises to be of much greater value for navigation, but which has not yet been fully developed as it was of little value in war time, is the wireless-acoustic method proposed by Professor Joly. In the original experiment of Collodon and Sturm in 1826, the velocity of sound in water was determined by striking an underwater bell and igniting a charge of gunpowder simultaneously. By knowing the distance from the source and observing the interval of time between the flash and the sound of the bell the velocity was determined, as light travels practically instantaneously over any ordinary distance. Conversely, if the time interval and the velocity are known, the distance of the source can be at once determined, as in the familiar method of ascertaining the distance of a lightning flash by noting the time between the flash and the thunder clap. The advantage of employing an underwater method is that sound is transmitted more effectively through water, and that there are no water currents comparable with winds to affect the velocity appreciably.

Unfortunately a flash of light is of no value in a fog, but wireless waves are little affected by it, and travel with the same speed as light, so that if a wireless flash and an underwater explosion are generated simultaneously at a lighthouse or other known position, and the ship is provided with a wireless equipment and a directional hydrophone, the distance of the station can be at once determined on the ship by noting the interval between the two impulses. As the velocity of sound in sea water is nearly a mile a second, the distance can be determined within a quarter of a mile by a simple stop-watch, and the direction of the source found by either the directional hydrophone or directional wireless, without any communication with the station. If the lighthouse or lightship simply sends out wireless impulses simultaneously with the strokes of the submarine bell at convenient intervals, all ships in the vicinity can locate their positions from time to time without delay or mutual interference, and if they are within the range of two such stations they can do so without any directional apparatus.

The recent developments in directional wireless have rendered the application of sound ranging to navigation of less importance, but even now wireless direction finding is not always reliable, especially at sunrise and sunset; and there is also liability to error on steel ships owing to their distorting effect on the wireless waves. As hydrophones become increasingly employed on ships for listening to submarine bells, &c., the ability to obtain accurate ranges by wireless acoustic signals will doubtless prove of great value.

Leader Gear

Although not strictly speaking an acoustic device, some mention should be made of the leader gear or pilot cables as an aid to navigation of harbours and channels in foggy weather. For this purpose it is necessary to be able to follow some well-defined track with a latitude of only a few yards, so that sound ranging is inadequate. But if a submarine cable carrying alternating current of sonic frequency, say 500 ~, is laid along the desired track, and the ship is provided with search coils with amplifier and telephones, the alternating magnetic field produced by the cable induces alternating electromotive forces in the coils, and thus gives a sound in the telephones when the ship is sufficiently near the cable. By using two inclined coils on the two sides of an iron or steel ship it is found that the sound is loudest when the telephones are connected to the coil which is nearer to the cable, so that the ship can be steered along it, and keep a fairly definite distance to one side of it, so that vessels passing in opposite directions will not collide. This device, which was first put forward by Mr. C. A. Stephenson of Edinburgh in 1893, was revived during the war by Captain J. Manson, and is now coming into use both in this country and in the United States. An 18-mile cable has been laid by the Admiralty from Portsmouth Harbour down Spithead and out to sea.

Acoustic Depth Sounding

Another purely acoustic device which promises to be of considerable value to navigation is that of depth sounding by acoustic echoes from the bottom. If a ship produces an explosion near the surface, the sound travels down to the bottom and is reflected back as an echo, and for each second of interval between the explosion and the echo the depth will be half the velocity of sound or 2500 ft., say 400 fathoms. Various experimenters, notably

M. Marti in France, Herr Behm in Germany, and Officers of the American Navy, have devised apparatus whereby the time between firing a detonator or other small charge under the ship and the reception of its echo from the bottom can be recorded on a high-speed chronograph, and very accurate results have been obtained.

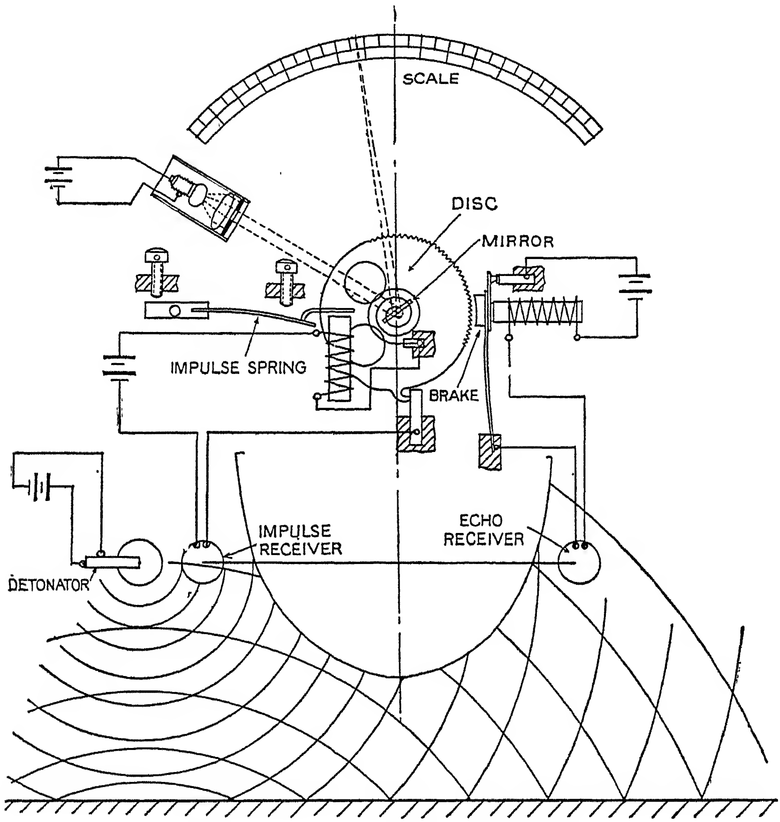


Fig. 29.—Behm's Acoustic Depth-sounding Method

The method of Behm, called the "Echolot" or echo-sounding device, now being developed by the Behm Echolot Co., Kiel, has attained a high degree of perfection, and is claimed to give indications in a ship at full speed, and even in rough weather, to an accuracy of within a foot. The transmitter consists of a tube through which a cartridge is impelled by air pressure into a holder fixed on the hull a little above the water line. The cartridge is fired out of the holder on pressing the firing-key, and is shot towards

the Impulse Receiver, while a time fuse in the cartridge is arranged to explode a detonator just before the cartridge reaches the microphone. Both the Impulse and Echo Receivers are microphones, but the latter is screened from the direct effect of the detonator by being fitted on the opposite side of the ship.

The explosion of the detonator causes a sudden drop in the current through the impulse receiver and weakens the current passing round an electromagnet, and causes it to release an "impulse spring" which suddenly starts a pivoted disc in rotation with a uniform velocity until the weakening of the current through the brake magnet, due to the echo reaching the echo microphone, stops the disc. The angular motion of the disc is therefore proportional to the interval between pressing the firing-key and return of the echo, and a light mirror on the disc spindle causes a spot of light to revolve round a translucent scale divided in depths, and to stop at the depth indicated. It is claimed that this timing device is capable of indicating short interval of time to an accuracy of one-ten-thousandth of a second, corresponding to only 3 in. in depth. Three keys are provided on the indicator, one for restoring the indicator to zero, one for firing the charge and obtaining the depth, and the third for checking the indicator against a standard time interval. A number of detonator charges can be stored in the transmitter magazine, and fired as required. The whole apparatus can be operated by a few dry cells, as the lamp is lit only at the moment of restoration, indication, or checking, and the colour of the light is varied at each operation to eliminate risk of mistake. It is stated that a rock with an upper surface of only 2 sq. metres in area is sufficient to give a correct indication.

The British Admiralty have recently developed a very simple and accurate echo sounding gear.

Echo Detection of Ships and Obstacles

By means of leader gear, sound ranging, and echo sounding navigation in fogs may be made much safer and more regular, but there still remains the great danger of collision in the open sea between ships, and especially with wrecks, rocks, and icebergs. As far as ships are concerned the difficulty is to some extent met already by signalling with sirens, but the curious blanketing and reflecting or refracting effect of fogs is a source of considerable confusion and danger. Underwater signalling does away with this difficulty almost entirely, and as hydrophone equipments become

more common the risk of collision between moving ships will rapidly diminish.

With a good directional hydrophone equipment an ordinary steamship can easily be detected and its direction determined up to a range of some miles merely by the noise of its engines. But in the case of wrecks, rocks, and icebergs, which emit no sound, the danger is still very great, and nothing but an echo method will detect them. Unfortunately this is a difficult matter, as a ship or small rock at a moderate distance is a very small target for an echo, so that the echo is of very small intensity, and it may quite easily be masked by bottom echoes. However, Fessenden, by the use of his powerful electromagnetic transmitter, succeeded as early as 1916 in obtaining echoes from distant obstacles, and by employing directional transmitting and receiving devices, which concentrate the sound in the desired direction, the strength of the echo can be increased, disturbances reduced, and the direction and approximate range of the obstacle determined. As early as 1912, just after the *Titanic* disaster, a proposal to employ echo detection for avoiding similar dangers was put forward by Mr. Lewis Richardson, and it may be hoped that this method will ultimately eliminate the last of the serious dangers of navigation.

Acoustic Transmission of Power

Before concluding this article, reference ought to be made to the wonderful achievements of M. Constantinesco, as showing the possibilities of what may be called acoustic engineering. For the purposes of underwater signalling the power transmitted, although large in comparison with what we have heretofore contemplated in connection with sound, rarely exceeds a hundred watts; and it has been left for M. Constantinesco boldly to envisage the possibility of transmitting large amounts of power by alternating pressures in water of sufficiently high frequency to be described as sound waves. For many years it has been customary to illustrate the phenomena of alternating electric currents by hydraulic analogies, and the present writer has even written a book in which such analogies have been used as a means of giving a complete theory of the subject; but the obvious possibility of using such alternating pressures in water for practical purposes was entirely missed until M. Constantinesco conceived it, and immediately the idea occurred it was evident that the whole of the theory was ready to hand from the

electrical analogies. In a surprisingly short time, therefore, M. Constantinesco has been able to devise generators, motors, and transformers capable of dealing with large amounts of power transmitted by hydraulic pipes in the form of acoustic waves of a frequency of about 50 ∼. The generator is, of course, simply a high-pressure reciprocating valveless pump, and the motor can be of similar construction, but by having three pistons with cranks at 120°, three-phase acoustic power can be generated and employed in the motors. The first commercial application of M. Constantinesco's devices has been to reciprocating rock drills and riveters, for which this method is especially suitable, as the reciprocating motion is obtained simply from a cylinder and pistol without any valves whatever, and the power is transmitted by a special form of flexible hydraulic hose pipe comparable with an electric cable. It is not too much to say that M. Constantinesco's ideas have opened up an entirely new field of engineering, and their development may have far-reaching effects.

For a discussion of the theory of hydraulic wave transmission of power, see Chapter VI.

Although this article is necessarily very incomplete, it will at least have served its purpose of showing the great importance of underwater acoustics, and there can be no doubt that a new department of scientific engineering has been opened up which has vast possibilities.

Developments in Echo Depth-sounding Gear.

Since the first appearance of this volume, the chief advance in underwater acoustic devices has been in the improvement of echo depth-sounding devices which have proved their great value for navigation and appear likely in time to become a standard feature of ship equipment. Three different types of such gear are now manufactured in this country: the Admiralty type by Messrs. H. Hughes & Sons; the Langevin piezo-electric type by the Marconi Sounding Device Company; and the Fathometer gear, which has been developed from the original Fessenden apparatus by the Submarine Signal Company. All these devices have now been made to give both a visual indication of the depth on a dial and a continuous record on a chart.

The basis of all methods of acoustic depth sounding is the recording of the time taken for a signal to travel from the ship to the bottom of the sea and return, but they differ in the type of the signal

and method of indication, and may be divided into impulse or "sonic" methods and high frequency or "supersonic" methods. In the former class to which the Behm "Echolot" (see p. 331), the original Admiralty sonic gear, and the Fathometer belong, the signal is in the form of a single powerful impulse provided by an explosive cartridge or an electromagnetic or pneumatic hammer striking a diaphragm; while in the latter a short train of high-frequency vibrations is emitted from a quartz piezo-electric oscillator, a steel rod which vibrates at a high frequency when struck by a hammer, or by a magnetostriction oscillator which is the magnetic analogue of the quartz oscillator.

The single impulse or sonic transmitter is practically non-directional, i.e. the disturbance travels equally in all directions under the ship. This has the advantage of making the indications practically independent of any rolling of the ship, but it has many disadvantages. Firstly, it is liable to give such a severe shock to the receiver at the moment the impulse is sent out that it does not recover in time to respond to an echo from a very shallow bottom; secondly, the greater part of the energy is wasted; thirdly, the echo must be very strong to be heard above the noises caused by the ship's machinery and motion through the water, and fourthly, it may not give true depths if the bottom is shelving steeply, as the first echo is received from the object which is nearest to the ship. With the high-frequency method the sound can be concentrated within a cone of any desired angle, so that the receiver can be fairly close to the transmitter without sustaining any severe initial shock, and the receiver can be sharply tuned to the transmitted frequency, so that it is nearly deaf to any other disturbances. If the ship could be kept on a perfectly even keel, the narrower the beam the better, as it would be equivalent to a vertical sounding line, but on account of rolling it is desirable that it should have an angle of something like half the maximum angle of roll. For a circular transmitter the semiangle of the beam θ is given by the relation $\sin\theta = 1{\cdot}2\dfrac{\lambda}{d}$ where λ is the wave length of the sound and d the diameter of the transmitter, so that we can obtain any beam angle we please by varying the diameter and frequency.

As regards receivers, a granular microphone is the most suitable for the single impulse or sonic system, and it must be mounted at some distance from the transmitter and preferably on the other side of the keel, so as to be shielded as much as possible from the initial

shock. This separation is however objectionable, as it seriously reduces the accuracy of sounding in very shallow water, where it is frequently most important. The device, of course, indicates the distance from the transmitter to the bottom and back to the receiver, and this varies very little when the depth under the keel is small compared with their separation. With the high-frequency system, however, the transmitter and receiver can be close together, so that this difficulty does not arise; and as both the quartz and magneto-striction transmitters will also serve as receivers, it is even possible to dispense with a separate receiver, as is done in the Marconi gear.

The essential function of the indicator is, of course, the measurement of the time interval between the impulse and echo. As the average velocity of propagation of sound in sea water is about 4900 ft. per second, and the sound has to travel the double distance to the bottom and back, each second of interval corresponds to a depth of 2450 ft. or about 400 fathoms; and if soundings are required within an accuracy of one foot, the time must be measured within an accuracy of four ten thousandths of a second. The most simple and reliable method of effecting this is by the contact method employed in the Admiralty sonic gear, in which the receiving earphones are shunted by two brushes, which press on a revolving ring which has a small gap in it, so that the phones are short-circuited for all but an interval of one or two thousandths of a second in each revolution. The transmitter is actuated at a certain moment in each revolution, and the two brushes are carried on an arm which can be turned by the observer until the short circuit is removed simultaneously with the arrival of the echo. The depth is then indicated by the position of the arm on a scale which can be divided in feet or fathoms. Direct visual indication is, of course, preferable, and is secured in the Fathometer gear by a revolving disc mounted close behind a ground-glass scale. The disc has a narrow slot in it, behind which is a small neon lamp, and the echo when sufficiently amplified, causes this lamp to flash and show a momentary red streak on the scale at each revolution. In the Marconi gear the amplified echo is received by an oscilloscope or high-frequency galvanometer the beam from which falls on an oscillating mirror and shows a luminous streak on a ground-glass scale. When the echo is received the momentary kick of the galvanometer shows as a kink in this luminous streak at the corresponding depth on the scale. Messrs. Hughes have produced a direct-reading pointer indicator for the high frequency Admiralty gear, which operates on the phase indicator principle. A revolving

solenoid is fed with direct current and therefore produces a rotating magnetic field, and a soft iron needle is momentarily magnetised by the current from the echo receiver, so that it sets itself along the axis of the solenoid at that moment.

Any of these devices enable the depth to be observed at intervals of every few seconds even when the ship is running at full speed, which is an enormous advantage over the old lead line, which required the ship to be running dead slow. Merely for ensuring safety in navigating shallow waters this is sufficient, but a great gain is secured by making the apparatus record the depths continuously on a chart which gives a profile of the bottom along its course, as this enables a ship to locate its position with considerable precision if the con-

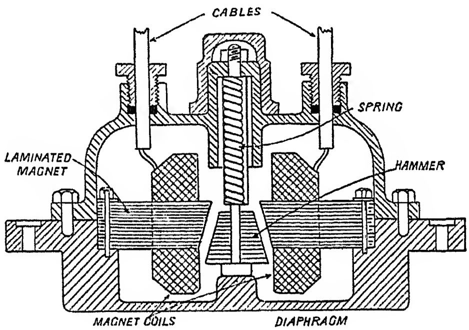


Fig. 30.—Electromagnetic Hammer Transmitter for Shallow Water Gear

tour of the bottom is accurately known. During the last few years recorders have come into general use, and have been found very satisfactory. The motor which actuates the transmitter contacts and the receiver mechanism is also employed to move a stylus uniformly across a band of paper which has been previously soaked in a sensitive solution (usually starch and potassium iodide, as in the early Bain printing telegraph), and the amplified and rectified echo causes it to make a mark on the paper at the moment it is received. The paper band is moved slowly forward at a constant rate by the same motor, or it can be driven from an electrical log so as to move proportionately to the distance covered by the ship, and, as the stylus makes a mark for each echo, a practically continuous line is drawn on the paper showing the variation of depth either with time or distance. By simple contact devices the stylus can also be made to mark the paper at each five or ten feet or fathoms of depth, and at regular intervals of time or distance, so that the record is complete,

and can be reproduced directly in a hydrographic atlas. Fig. 31 shows such a record of a 15 minutes' run, with a shallow water magnetostriction set.

After the above general description, the only features of the various gears which require special consideration are the trans-

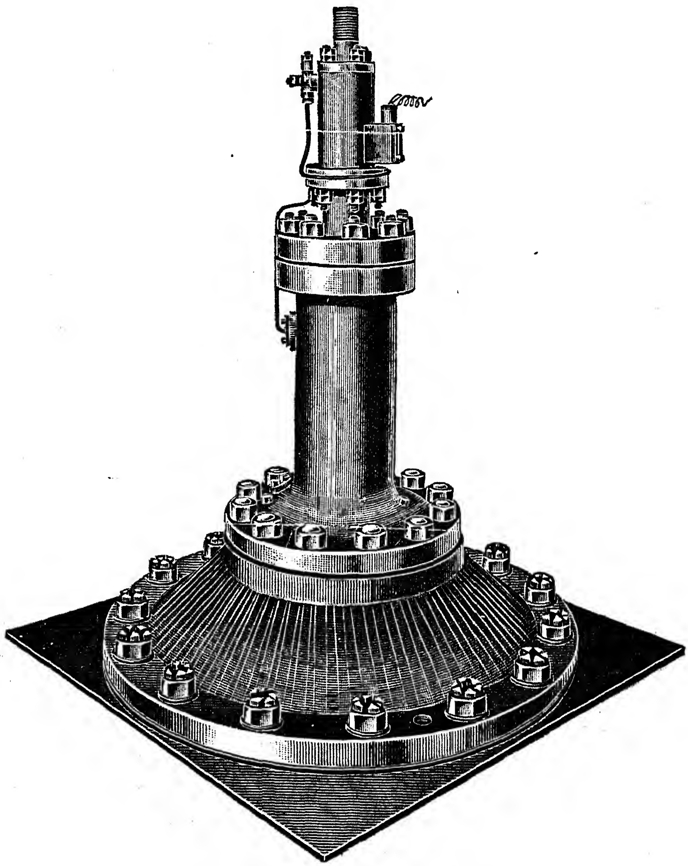
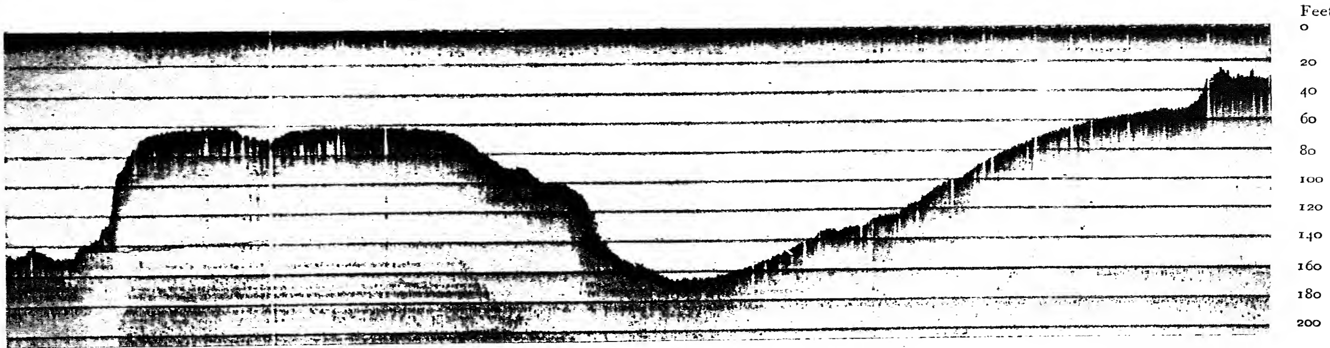


Fig. 32.—Pneumatic Hammer Transmitter

mitters. For the impulse or sonic transmitters the types employed in the shallow water Admiralty gear and the Fathometer gear are very similar, and the former is shown in fig. 30. A ring of iron stampings, with internally projecting poles, is excited by coils on the poles, and the hammer consists of a tapered block of stampings, which is drawn into the gap between the poles and compresses a spiral spring which drives the hammer down against a diaphragm when the



H. Hughes & Son, Ltd.

FIG. 31.—DEPTH RECORD

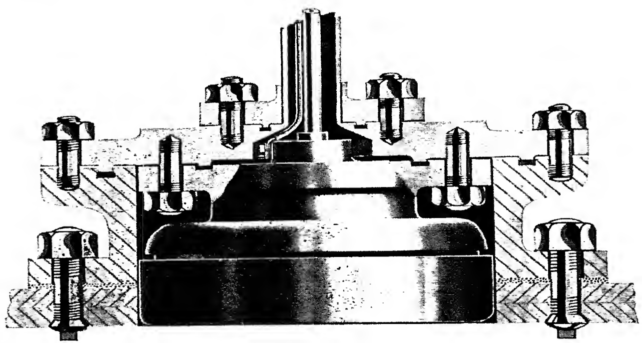


FIG. 33.—MARCONI QUARTZ TRANSMITTER

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current is broken. For the deep water Admiralty gear, which has been used for depths of over 2000 fathoms, the hammer is operated pneumatically with an electromagnetic release (fig. 32).

The Marconi high-frequency quartz transmitter, which also serves as the receiver, shown in figs. 33 (p. 338) and 34, has a thin layer of quartz crystals H cemented between two steel discs F and G,

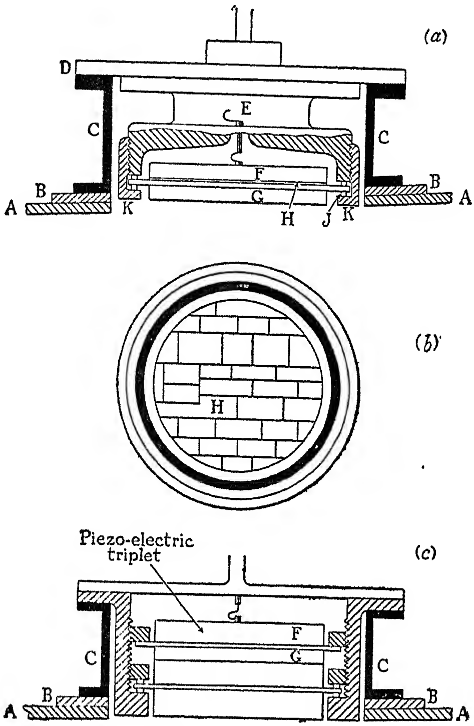

Fig. 34.—Marconi Quartz Transmitter

the lower of which is usually in contact with the water while the upper is highly insulated and connected to a high-voltage oscillator which gives a frequency of about 37,500 periods per second, producing waves about 4 cm. long in the water, and a somewhat sharp beam. In order to provide for the removal and replacement of the oscillator without dry-docking the ship, the housing is sometimes provided with a second resonant steel plate shown at the bottom of C which is clamped by a central flange and transmits the oscillations to the water.*

* Cdr. J. A. Slee, C.B.E., R.N., *Journal Institution Electrical Engineers*, Dec. 1931.

Quartz oscillators, although highly efficient, are somewhat costly and require special technique in construction, and hence efforts have been made to obtain a high-frequency impulse without employing crystals. One simple method, which is fairly effective, is to employ the ordinary hammer of the single-impulse transmitter, but to substitute a steel rod clamped at its centre like the lower plate in the Marconi transmitter for the diaphragm. This rod

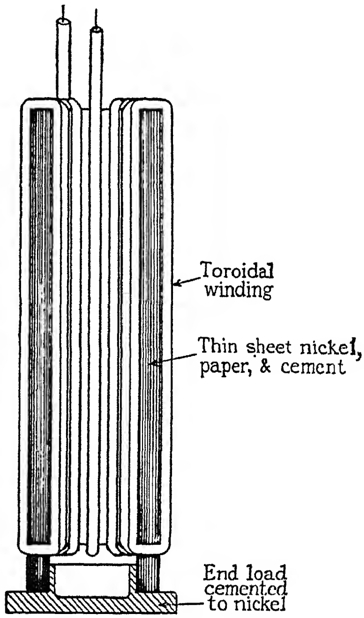

Fig. 35.—Magnetostriction Scroll-type Oscillator

vibrates with its resonance frequency and emits a short train of damped oscillations each time it is struck.

But within the last few years a great advance has been made by employing the principle of magnetostriction, i.e. the deformation which takes place in magnetic materials when they are magnetized. This effect is most marked in pure nickel, and in certain nickel and cobalt steel alloys, and it enables oscillators of any frequency to be constructed very cheaply and by ordinary workshop methods. It lends itself to very various forms of oscillator, but the two which have been found most convenient for echo sounding work are the "scroll" and "ring" types shown in figs. 35 and 36. In the former, a strip of nickel is simply wound up like a scroll of paper, and is

provided with a simple toroidal winding like a gramme ring armature. When this winding is supplied with alternating current the scroll expands and contracts axially, so that a disc cemented to one end serves as the emitting surface. The axial length of the scroll is made such that its mechanical resonance frequency is that required

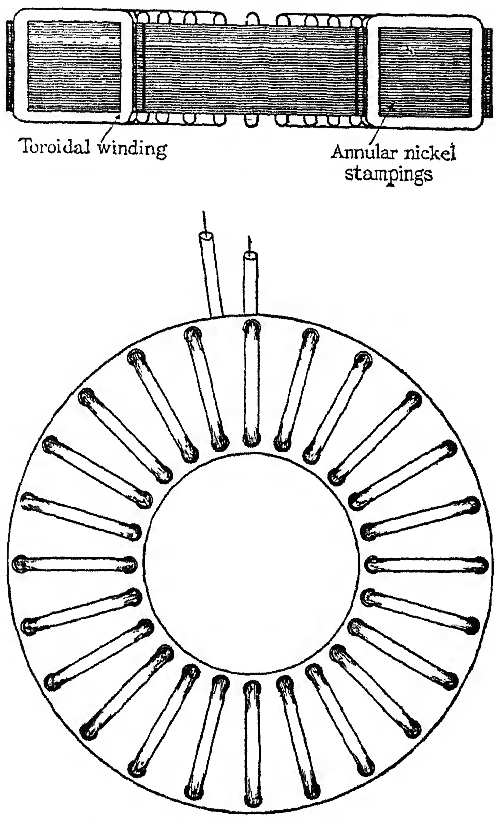

Fig. 36.—Magnetostriction Ring Oscillator

(usually about 15,000 periods per second), and the winding is fed with alternating current at a low voltage either from a valve oscillator or a condenser discharging through an inductance, in either case at the resonant frequency. In the disc type the oscillator is made up of a number of ring stampings with holes round its inner and outer periphery. These stampings are cemented together into a solid block with an insulating cement, and a toroidal winding is wound through the holes. When supplied with alternating current the

ring expands and contracts radially, and it is operated at its resonance frequency. As the sound is emitted in all directions in a plane parallel to the ends of the ring, it is surrounded by a sound reflector made of two thin metal cones with india-rubber " mousse " between

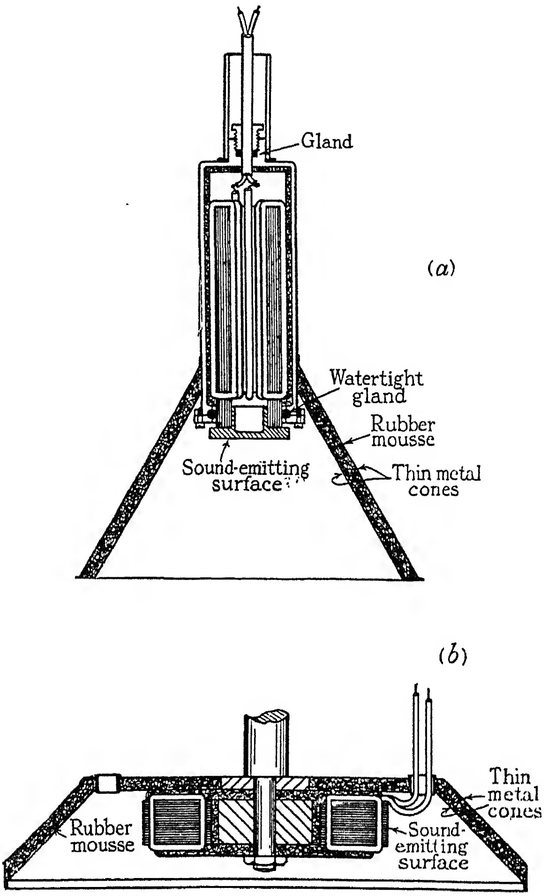

Fig. 37.—Magnetostriction Oscillators and Reflectors

them, as shown in fig. 37, and the beam angle can be varied by choosing the diameter of this reflector. The transmitters will serve equally well as receivers, but it is found preferable to mount two of them close together, one acting as transmitter and the other as receiver.

These magnetostriction transmitters and the associated recorders were designed by Drs. A. B. Wood and F. D. Smith and Mr. J. A.

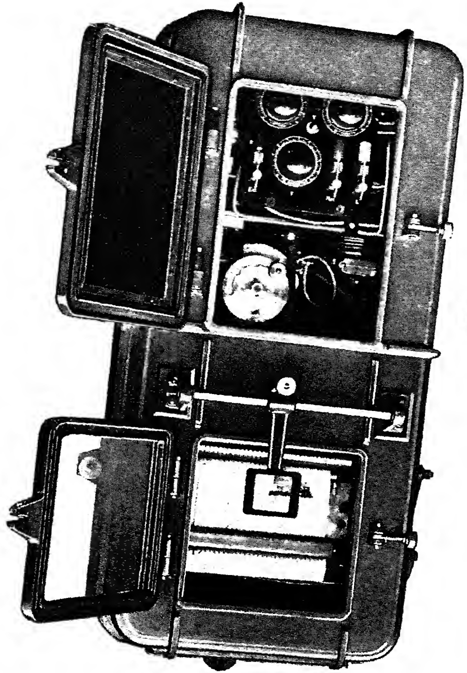

FIG. 39.—MARCONI INDICATOR AND RECORDER

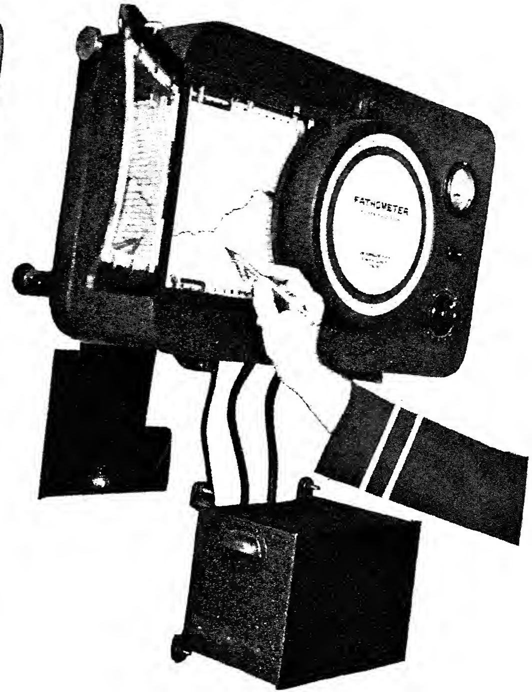

FIG. 40.—FATHOMETER INDICATOR AND RECORDER

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McGeachy,* of the Admiralty Research Laboratory, after a research on magnetostriction by Dr. E. P. Harrison, and have been incorporated with great success in the latest forms of Admiralty depth-

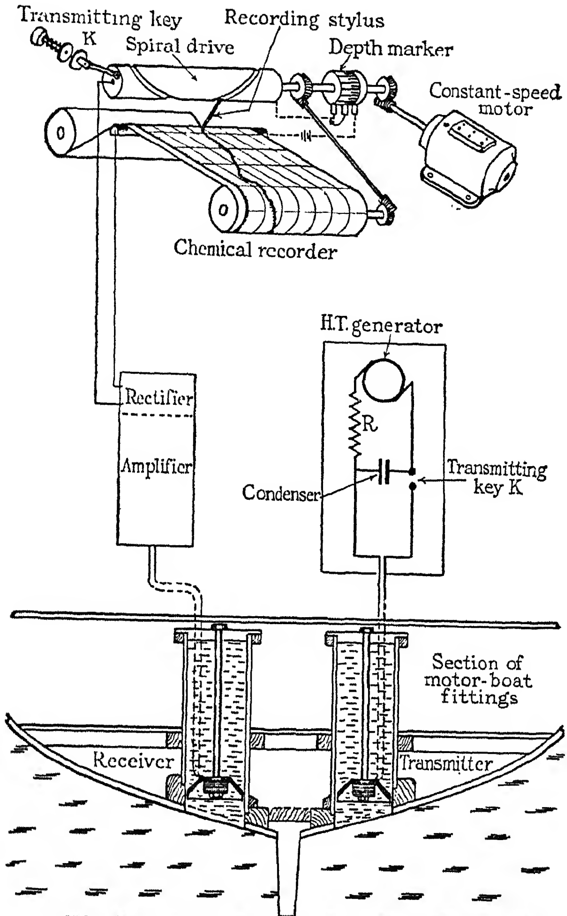


Fig. 38.—Magnetostriction General Arrangement

sounding gear. As they only require very small power for shallow depths without high-voltage oscillators, it has been possible to instal them with recorders in small motor-boats for the hydrographic survey of shallow rivers and estuaries, which has enormously increased the facility and rapidity of such surveys. On the other hand, they have

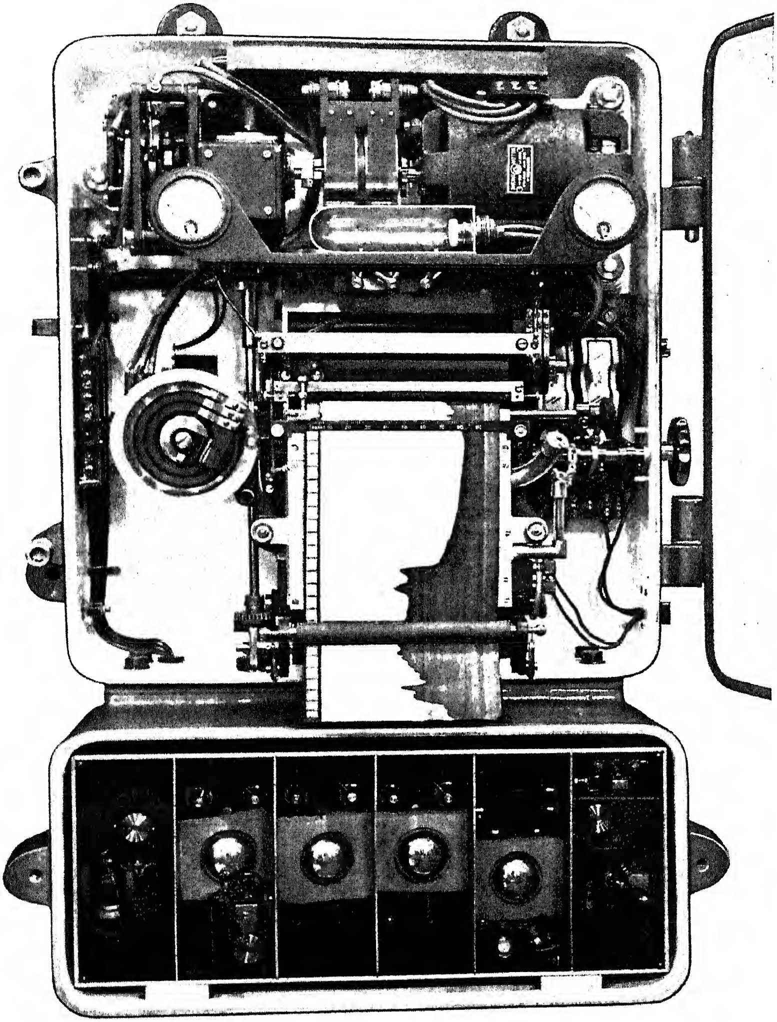
* *Journal Institution of Electrical Engineers*, Vol. 76, No. 461, May, 1935, p. 550.

proved equally efficient for moderate depths and deep water gear, and soundings have been taken successfully in depths of 2000 fathoms with the transmitters and receivers in water-filled tanks in contact with the hull, and transmitting and receiving through the ship's plates. Fig. 38 is a diagram of the motor-boat outfit with chemical recorder.

Figs. 39, 40 and 41 show external views of the Marconi, Fathometer, and Admiralty indicating and recording sets (Plates facing pp. 342, 346).

Probably over two thousand naval and mercantile vessels have by now been equipped with echo depth-sounding gear of one or other of the above types, and the reports on them show their great value, accuracy and reliability. For moderate depths down to 200 or 300 fathoms, an accuracy of a foot is obtainable, while the magnetostriction motor-boat gear actually records depths to an accuracy of three inches even when the boat is almost touching bottom. The deep water set on the *Discovery II* was so satisfactory in Antarctic waters that line sounding was discontinued, and such sets have been of great service in cable-laying ships by enabling them to lay their cables on the least irregular bottoms. The importance of continuous and recorded soundings for mercantile ships by facilitating their entry to harbours and locating their positions at sea has already been referred to, and its value will increase as more and more records along the main trade routes become available. Lastly, a remarkable application of such gear has been found for fishing, and many trawlers are now being equipped with it, as it has been found that the quantity and quality of catches depends greatly on the depth, and echo sounding gear enables the ship to follow a contour line of the depth desired. Echo depth-sounding has proved the most useful application of underwater acoustics and seems likely to be universally adopted.

The writer is indebted to the three firms above mentioned for particulars and illustrations in this section, and to the *Journal of the Institution of Electrical Engineers* for figs. 34, 35, 36, 37, and 38.



H. Hughes & Son, Ltd.

FIG. 41.—ADMIRALTY SUPERSONIC RECORDER

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CHAPTER X

The Reaction of the Air to Artillery Projectiles

Introduction

All calculations of the motion of a projectile through the air are directed to one object—to determine the position and velocity of the projectile at any given time after projection in any prescribed manner. In general the reaction of the air to a rotating projectile is very complicated; the complication is considerably reduced, however, if the projectile can be made to travel with its axis of symmetry coincident with the direction of the motion of its centre of gravity. It is a matter of experience that by giving the projectile a suitable spin it can be made to travel approximately in this manner for considerable distances; in such circumstances the reaction of the air is reduced to a single force, called the *drag*, which acts along the axis of the projectile and tends to retard its motion.* When this drag is known for a projectile of given size and shape the problem enunciated above becomes one of particle dynamics, and its solution for that projectile can be effected, at all events, numerically. The first and major part of this chapter is devoted to the consideration of this drag.

When the angle of elevation of the gun is considerable the curvature of the trajectory increases too rapidly for these simple conditions to hold. The motion then becomes complicated and the problem becomes one of rigid dynamics in three dimensions; the trajectory is a twisted curve instead of a plane one, and the well-known phenomenon of *drift* appears. Similar complications arise when the projectile is not projected with its axis coincident with the direction of motion, or when the spin is insufficient to maintain this coincidence.

* A couple of small magnitude due to skin friction also exists; it acts about the axis and tends to reduce the spin; its effect is generally negligible with modern projectiles.

In the second part of this chapter the component forces and couples of the reaction of the air in these circumstances are briefly considered.

THE DRAG

Early Experiments: the Ballistic Pendulum

Most early writers on ballistics* assumed that the resistance of the air (the drag) to the motion of projectiles was inconsiderable. The first experimenter to attempt the determination of the air drag on projectiles moving at a considerable speed was Robins, who, in 1742, carried out experiments with his ballistic pendulum. He found that the resistance encountered was abnormally greater for velocities greater than about 1100 ft. per second than for lesser velocities. Following Robins, many experiments were performed with the ballistic pendulum, notably at Woolwich (by Hutton, 1775–88) and Metz (by Didion, 1839–40), to determine the drag as a function of the velocity of the projectile.†

The method employed by Robins was, briefly, as follows:

A gun was placed at a known distance from a heavy ballistic pendulum; the charge was carefully weighed and the projectile was fired horizontally at the pendulum. The latter received the projectile in a suitable block of wood, and the angle through which it swung was recorded. Knowing the weights of the projectile and pendulum and the free period of oscillation of the latter, the velocity of the projectile at the moment of hitting could be calculated. The experiment was repeated with the same charge, the distance between the pendulum and gun being varied from round to round. There resulted a series of values of velocity at known distances from the gun; the retardation of the projectile and hence the resistance of the air at these distances could be deduced.

By performing similar sets of experiments with various weights of charge, the drag could be determined as a function of the velocity of the projectile.

The uncertainty of realizing the same muzzle velocity in each set of experiments with constant charge vitiated the reliability of the results. Hutton overcame this difficulty by hanging the gun hori-

* The study of the flight of projectiles.

† For a full account of these experiments see: Robins, *New Principles of Gunnery*, 1761; Hutton, *Tracts*, 1812, especially Tract XXXIV; Didion, *Lois de la résistance de l'air* (Paris, 1857).

zontally from a suitable support, so that the gun itself became the bob of another pendulum. From the angle through which this system swung on firing, the muzzle velocity of the projectile could be calculated. For each round fired he thus obtained two values of the velocity—one at the muzzle, the other at a known distance from the muzzle.

Let v_1 and v_2 be these values, and let x be the distance between gun and pendulum. Then, if m be the mass of the projectile, the loss of energy in traversing the distance x is $\frac{m}{2}\left(v_1^2 - v_2^2\right)$. If R be the mean value of the drag we therefore have

$$R = \frac{m}{2x}\left(v_1^2 - v_2^2\right).$$

Provided that the distance x is sufficiently small, this value may be taken as the actual value of the drag for the velocity $v = \frac{1}{2}(v_1 + v_2)$.

By varying the charge and the distance between the gun and pendulum Hutton determined the drag numerically as a function of the velocity.

From the time of Hutton to the present day, experiments conducted on the Continent and in America to measure the resistance of the air have been based on this principle, namely, to determine the velocity at two points on an approximately horizontal trajectory a known distance apart. A large number of instruments for measuring the velocity of a projectile at a given point have been invented during this time; the reader is referred to *Balística Experimental y Aplicada*, by Col. Negrotto of the Spanish army (Madrid, 1920), for an up-to-date and exhaustive account of them. It should be noted here that few of these chronographs were invented especially for the determination of the resistance of the air; there are, of course, many important uses for such instruments in gunnery.

Since 1865 experiments on the resistance of the air conducted in England have been based on a different principle. The method was first proposed by the Rev. F. Bashforth, B.D., sometime Professor of Mathematics at the Artillery College, Woolwich; it consists in measuring the times at which a projectile passes a number of equidistant points along an approximately horizontal trajectory. These times are then smoothed and differenced, and the velocity and retardation of the projectile at a number of corresponding points are calculated by the method of finite differences. This method is

evidently more economical in expenditure of ammunition than that of foreign experimenters.

The Bashforth Chronograph

In 1865 Bashforth invented his now-famous electric chronograph,* by means of which he succeeded in measuring small intervals of time with an accuracy previously unattained in ballistic instruments.

The chronograph consists essentially of two electro-magnets, to the keepers of which two scribers are attached by linkwork; these scribers trace continuous spiral lines on paper fixed on a revolving cylinder. The two spirals are generated by a mechanical movement of the framework supporting the electro-magnets in a direction parallel to the axis of the cylinder. The movement of each keeper is controlled by a suitable spring, and any small movement of either is identified on the record by a kink on the corresponding spiral trace.

One of the electro-magnets is connected with a clock and the current is broken momentarily every second; one of the spiral traces thus constitutes a time record. The other electro-magnet is connected in series with screens placed at equal distances along the trajectory of the projectile. At the moment the latter passes a screen the current is broken for a short interval of time and a kink is made in the corresponding spiral trace. The mechanism by which the current is broken is as follows.

A board is supported in a horizontal position with its length at right angles to the direction of motion of the projectile. Transverse grooves are cut in the board at equal distances, somewhat less than the diameter of the projectile. Hard spring-wire staples are fixed in the board so that each prong projects upwards from a groove.

On the near edge of the board a number of copper straps are fixed; each strap has two oval-shaped holes which are placed at the near ends of adjacent grooves. The prongs of the staples are bent down into the grooves and project through the oval-shaped holes; the arrangement is such that the butts of the staples and the copper straps alternate, so that a current may pass continuously through the staples and straps.

The prongs terminate in hooks from which are suspended small

* For a full account of Bashforth's experiments consult his *Revised Account of the Experiments made with the Bashforth Chronograph* (Cambridge University Press, 1890).

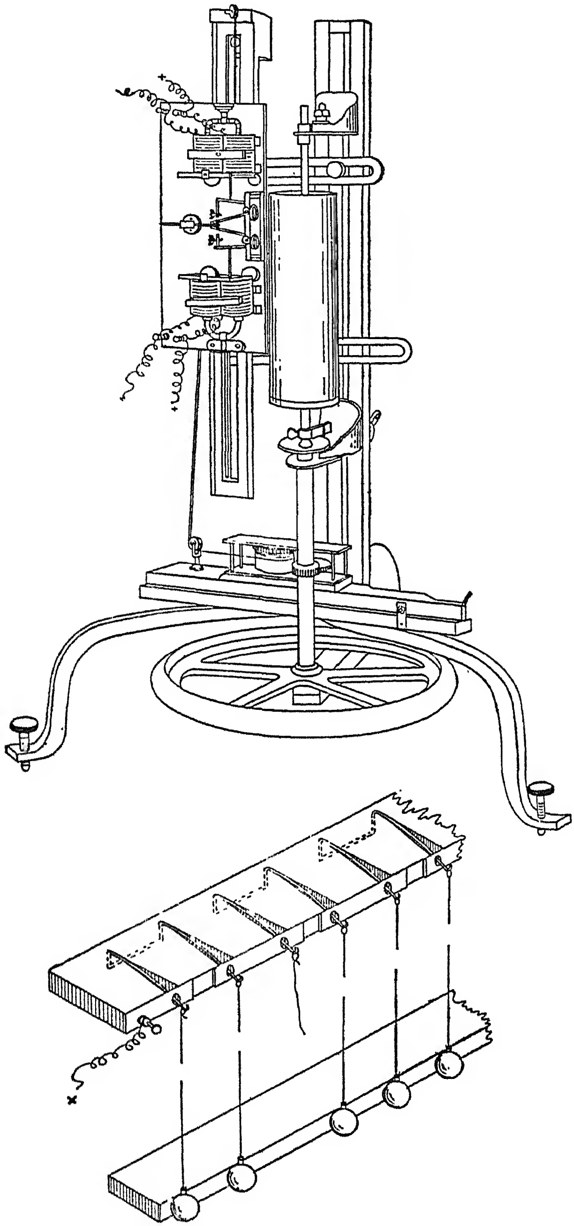


Fig. 1.—The Bashforth Chronograph and Screen. Reproduced by courtesy from "Description of a Chronograph", by F. Bashforth, B.D., *Proceedings of the Royal Artillery Institution*, 5, 1867.

equal weights by means of fine cotton; the weights rest against a second horizontal board supported some distance below the first, and are sufficiently heavy to maintain the prongs in contact with the bottom edges of the holes in the straps.

When the projectile passes it will break at least one of the cottons; the corresponding prong will spring from the bottom to the top of its hole in the copper strap and so break the current momentarily. This mechanism constitutes a " screen ".

The record, when removed from the cylinder and laid flat, consists of two parallel straight lines with kinks in them; in the upper line the kinks correspond to the passage of the projectile through successive screens; in the lower the kinks indicate seconds of time. With a suitable measuring apparatus it is possible to read off the time intervals between the screens to four decimal places of a second.

Bashforth continued his experiments until 1880, and produced a table giving values of the air drag for velocities up to 2780 ft. per second. Contemporary experiments were also conducted in Europe by Mayewski (Russia), Krupp (Germany), and Hojel (Holland), giving results in substantial agreement with those of Bashforth.

Later Experiments

In the early years of the present century a large amount of work was done in England, France, and Germany to obtain accurate information concerning the air drag. In 1906 the Ordnance Board used a method similar to that of Bashforth, but having a more accurate timing and recording device; the drag for velocities up to 4000 ft. per second was determined. In 1912 O. von Eberhard, at Krupp's,* made a large number of experiments with projectiles of various shapes and sizes. It is thought that these form the most exhaustive set of experiments yet undertaken; the results, which are frequently used in this chapter, are certainly the most complete yet published openly.

In this method the velocity at two points on the trajectory was measured by means of a spark chronograph; the distance between the points varied from 50 m. for small projectiles to 3 Km. for those of large calibre. The method of deducing the resistance was similar to that used by Hutton, and results for velocities up to 1300 m. per second were obtained.

* Cf. O. von Eberhard, *Artilleristische Monatshefte*, 69 (Berlin, 1912).

Krupp's 1912 Experiments

The velocity at a given point is measured, in this method, by firing the projectile through two screens; one screen is placed a measured distance (a few metres) in front of the point, and the other the same distance behind it. The spark chronograph measures the time taken by the projectile to traverse the distance between the screens; the average velocity between them is deduced and is taken to be the actual velocity at the point. The distance being short no appreciable error arises from this assumption.

Each screen consists of a square wooden frame across which fine copper wire is stretched backwards and forwards continuously in

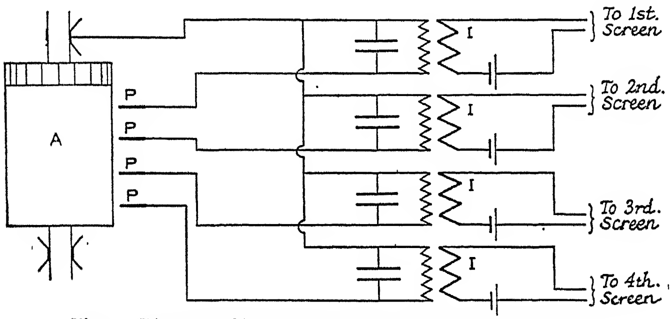


Fig. 2.—The Spark Chronograph used in Krupp's 1912 Experiments

such a way that a projectile passing through is certain to break the wire. In the experiments four screens are used; one pair serves to measure the velocity at the beginning of the measured range, the other, the velocity at the end of it.

The spark chronograph is shown diagrammatically in fig. 2. A metal drum A is rotated at high speed by means of a suitable motor, the speed being recorded by means of a Frahm tachymeter; readings to within one revolution per second can be taken with this instrument. It is essential that the speed of the drum be constant during the flight of the projectile over the range, and this instrument serves the additional purpose of indicating the most suitable moment to fire the gun. The surface of the drum is silvered and is coated with soot except at one edge where a circumferential scale is fixed.

There are four induction coils, I; their primary circuits, which contain batteries, are connected respectively to the four screens; one terminal of each secondary circuit is connected to the spindle of the drum, the other terminals being connected to sharp platinum

points, P. A break in one of the primary circuits will cause a spark to pass across the small gap between the corresponding point and the drum; the spark is enhanced by a condenser in parallel with the secondary circuit. The mark on the drum made by the spark is like a bright pin-point and is surrounded by a sort of halo; it is thus easily identified.

The positions of the marks are read by means of a microscope mounted on the frame supporting the drum; this microscope can be traversed parallel to the axis of the drum. To take a reading the drum is rotated by hand until the mark made by a spark is in the field of the microscope; it is then clamped. The mark is then brought to the zero line of the eyepiece by means of a fine adjustment. The microscope is then traversed to the edge of the drum and the reading is taken from the circumferential scale. The positions of the marks made by the other sparks are similarly measured. The time intervals between the pairs of sparks are then deduced with the aid of the tachymeter reading.

The chronograph is calibrated by breaking all the primary circuits simultaneously and recording the relative positions of the marks made by the sparks.

With this instrument such a small time interval as 0·0017 sec. can be measured with a probable error of $7{\cdot}5 \times 10^{-6}$ sec., or 0·44 per cent.

Experiments prior to the war thus fell into two types—the Hutton type, in which the velocity of a projectile was measured at two points a known distance apart, and the Bashforth type, in which the times of passing a number of equidistant points along the trajectory were recorded.

With regard to the first type, unavoidable errors in the measurement of the velocity would vitiate the results if the distance between the points were too short; on the other hand the approximate method of deducing the resistance as a function of the velocity cannot give satisfactory results unless the distance is short. It would thus appear to be a difficult matter to choose suitable distances, and laws of resistance based on methods of this type must be somewhat uncertain.

With regard to the second type, it is evident that, provided the time readings are sufficiently smooth to ensure that differences of finite order vanish, the resistance and velocity at corresponding points can be deduced to a known degree of accuracy. When, however, the observed times require appreciable alteration to make

them smooth, considerable uncertainty attaches to the results deduced.

To both types there is the objection that no account is taken of possible yaw * of the projectile. It is well known that all shells have some yaw on leaving the muzzle of the gun, and it cannot be hoped that it is always damped out sufficiently before reaching the points at which observations are made. At high velocities very considerable yaw may develop, and, in particular, such obstacles as screens may tend to increase it. In any case the yaw does not remain constant during the flight of the projectile, hence, unless it is at all times negligible, the resulting law of resistance cannot be consistent.

In any method which depends on observations of a projectile in flight it is therefore necessary to make some provision for observing the yaw as well as the velocity (or time) at points on the trajectory. If the yaw is small throughout the flight reliable results would be obtained; in cases of considerable yaw a method would have to be devised of correcting for it in the analysis of the records before any reliance could be placed on the deduced values of the resistance.

Cranz's Ballistic Kinematograph

A method in which provision is made for observing the yaw, at least qualitatively, was devised by C. Cranz,† who carried out experiments which were contemporary with those of Eberhard; it was, however, applicable only to rifle bullets and to similar projectiles of very small calibre.

A series of shadow photographs of the bullet was taken by means of his Ballistic Kinematograph at each end of a 20-m. range. The velocity at each end could be deduced from the positions of the successive images on the kinematograph film and the observed speed at which the film moved through the camera. The occurrence of appreciable yaw could be detected at once from the photographs, and the reliability of the records for the purpose of the experiments could thus be estimated.

The Solenoid Method

Since the War technique has developed considerably in the measurement of high velocity. A very successful method, developed by Sir Frank Smith, of measuring time intervals in experiments of

* The yaw is the angle between the axis of the shell and the direction of motion of its centre of gravity.

† For a full account of Cranz's experiments see *Artilleristische Monatshefte*, 69 (Berlin, 1912).

the Bashforth type, consists in firing an axially-magnetized projectile through the centres of a series of equidistant solenoids which are connected in a series with a sensitive galvanometer. The current induced in each solenoid reaches a maximum as the projectile approaches, falls rapidly to a minimum as the projectile passes through, and finally returns to its original value as the projectile emerges. The " signature " of the galvanometer is recorded photographically on a rapidly moving film on which is also recorded the oscillations of a tuning-fork of known frequency.

Experiments with High-velocity Air Stream

During the war experiments were undertaken in a new direction. Instead of making observations on a projectile in flight, the thrust on a stationary projectile in a current of air moving at high velocity was directly measured.* The method has subsequently undergone considerable development in France and America,† and at the National Physical Laboratory.‡

The projectile (or a scale model) is supported by means of a thin steel spindle fixed to the centre of the base in prolongation of its axis; this spindle is attached at its other end to a mechanism designed to measure the thrust on the projectile. Compressed air issues from a reservoir through a suitable orifice, thus generating a high-velocity stream; the projectile is placed in the centre of the stream. The determination of the most suitable size and shape of the orifice was a matter of considerable difficulty; after a large number of trials an orifice was obtained which ensured a steady stream in the vicinity of the projectile.

The temperature and velocity of the air in the stream are computed on the classical theory from the state of the air in the reservoir before the orifice is opened. When the velocity is greater than that of sound in air a check on the computed value can be obtained by photographing the head wave caused by the projectile (see p. 357) and measuring its slope.

The possibilities of such an experimental method are innumerable. Apart from the direct determination of resistance of air at all angles of yaw, its sphere of usefulness extends to the elucidation of many problems connected with the general reaction of the air on projectiles.

* Experiments of various kinds on projectiles had previously been carried out in wind channels, but the velocity of the air stream was at most 30 m. per second.

† See " The Experimental Determination of the Forces on a Projectile ", by G. F. Hull, *Army Ordnance*, Washington, May–June, 1921.

‡ See the Annual Reports of the Director, N.P.L. for 1922 and succeeding years.

Considerable progress has been made in experimental ballistics since the war. The time, the yaw, and the orientation of the axis of the projectile at a series of points along a horizontal trajectory can be observed with considerable accuracy; results from such experiments co-ordinated with those of experiments with the high-velocity air stream, have placed our knowledge of the reaction of the air on a projectile upon a very sound foundation.

The Drag at Zero Yaw

The resistance of the air to a projectile of given shape, moving with its axis of figure coincident with its direction of motion, depends on the following arguments:

the velocity, v, of the projectile;
the calibre (i.e. diameter), d;
the characteristic properties of the air, chiefly the density, the the elasticity, and the viscosity.

Dimensional considerations lead us to the form

$$R = \rho d^2 v^2 f(v/a, vd/\nu) \quad \ldots\ldots\ldots\ldots \quad (1)$$

for the resistance R of the air to projectiles of given shape, where ρ is the density of the air, a is the velocity of sound in air, an index of the elasticity, and ν is the kinematic viscosity. The function $f(v/a, vd/\nu)$ in this expression is called the *drag coefficient*. Since $\rho d^2 v^2$ has the dimensions of a force it follows from equation (1) that the drag coefficient has no physical dimensions; its arguments must therefore be so chosen that they shall have no dimensions; v/a and vd/ν both satisfy this condition, and are the simplest arguments in terms of which the function can be expressed.

The Drag at Low Velocities

For velocities below the critical velocity it is well known that

$$f(v/a, vd/\nu) = A\nu/vd$$

where A is a constant, the terms in v/a being negligible. This leads to the expression

$$R = A\rho d\nu v$$

for viscous drag.

For velocities higher than the critical velocity we have a change of physical conditions; the air behind the projectile breaks up into

eddies and the linear law of viscous drag no longer holds.* In such circumstances the resistance is found experimentally to be approximately proportional to the square of the velocity.

For incompressible fluids an expression of the form

$$A\nu/vd + B$$

for the drag coefficient will usually fit experimental data for bodies completely immersed, A and B being constants depending on the shape of the body. When v is very small the first term is large compared with the second, and the linear law for viscous drag reappears; when, on the other hand, v is large the first term becomes small compared with the second, and we have an approximate quadratic law.

An expression of the same form will also hold for the resistance of air to a projectile, provided the velocity is not sufficiently high to cause compression of appreciable amplitude. No upper limit of velocity can be fixed for this law, since the amplitude of the compression will depend on the shape of the head of the projectile as well as on the velocity; thus, for Krupp normal shells, which have a more-or-less pointed head, the drag coefficient changes extremely slowly with v even at a velocity of 215 m. per second, showing that the approximate quadratic law holds for these projectiles at this velocity, whereas with cylindrical projectiles (flat heads) the drag coefficient changes rapidly at this velocity (see fig. 5).

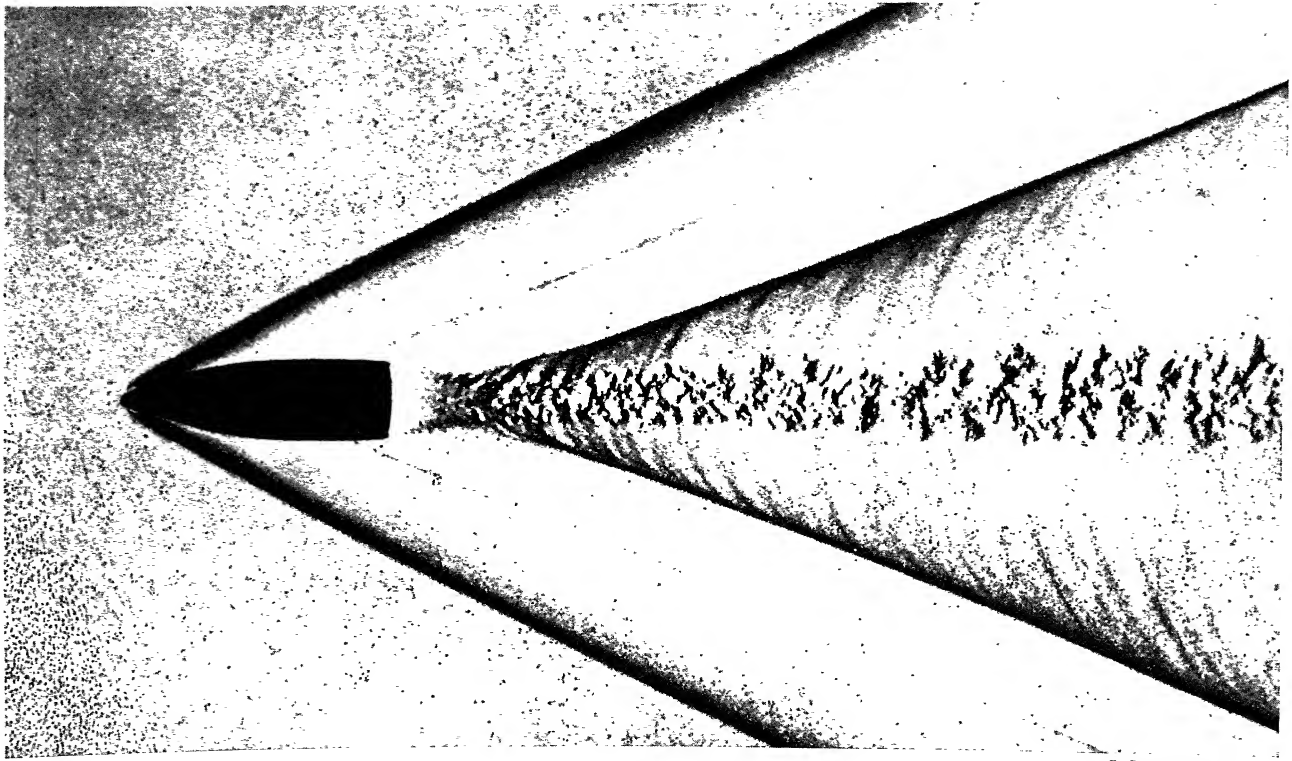
There is little experimental evidence of the behaviour of the drag with variations of d for projectiles at these velocities. The results of wind-channel experiments confirm the form given above for the drag coefficient for velocities up to 30 m. per second, and it has generally been found that the drag coefficient is greater for projectiles of small calibre than for those of large calibre of the same shape.

The Drag at High Velocities

For velocities greater than the velocity of sound the physical conditions are again changed.

At the nose of the projectile the air undergoes condensation. The air being an elastic fluid, a condensation formed at any point in it is transmitted in all directions with a velocity which is, in general, the *velocity of sound*. If, then, the projectile is travelling with a velocity less than that of sound, the condensation of the air at the

* See Chapter III, p. 103.



C. Cranz (Berlin, 1917)

FIG. 3.—PHOTOGRAPH OF AN 8-MM. BULLET MOVING WITH VELOCITY OF ABOUT 880 M./S.

Facing page 356

nose will be transmitted, as soon as it is formed, away from the nose in all directions. If, on the other hand, the projectile is moving faster than sound is propagated, the condensation of the air at the nose cannot be transmitted away from the nose in all directions; it can be transmitted away laterally, but not forwards. The result is that the nose is always in contact with a cushion of compressed air. Greatly increased pressure is thus experienced by the projectile when travelling with a velocity greater than that of sound.

Photographs of bullets moving with such velocities reveal the existence of two wave fronts, somewhat conical in shape, one at the head and the other at the base. In fig. 3 a photograph taken by Cranz in his ballistic laboratory is reproduced.

The wave front at the head can be accounted for by Huyghens' principle; it is in fact the envelope of spherical waves which originate at the head of the projectile at successive instants of time. If the amplitude of the condensation, when first formed, were small the wave front would be a cone of semi-angle Ω, such that

$$\sin\Omega = a/v. \qquad \text{(2)}$$

When v is less than a the spherical waves have no envelope, and, of course, no wave front is formed.

In the actual state of affairs the amplitude of the condensation at the nose is not small, but finite. The velocity at which it is propagated is therefore greater than the normal velocity of sound. At points on the wave front near the nose we should therefore expect the angle Ω to be greater than at more distant points where the amplitude has become considerably reduced. The form of the actual wave front at the nose is therefore a blunted cone, and the flatter the head of the projectile the more is the wave front blunted.

The formation of the waves behind the projectile cannot be accounted for in such a satisfactory manner. Lord Rayleigh * has shown that the only kind of wave of finite amplitude which can be maintained is one of condensation; his argument refers to motion in one dimension only, but we see no reason for modifying the result when applied to motion in three dimensions. We therefore conclude that the wave at the base of the projectile is, like the head wave, one of condensation. An examination of the photographs of bullets in flight verifies this conclusion.

* "Aerial Plane Waves of Finite Amplitude", *Scientific Papers*, Vol. V.

The source of disturbance causing this wave might be identified with the relatively high state of condensation of the air flowing into the rarefied region at the base.

The angle Ω of the straight part of the wave appears generally to be less than that of the head wave; the difference in angle is probably the geometrical consequence of placing the source of light close to the bullet; we have, in fact, a perspective view of the waves. When the source of light is very close to the bullet the consequent difference in angle may be considerable.

The tendency of the angle to diminish towards the apex of the wave is probably due to two effects. In the first place there may be some variation in temperature of the air in the immediate neighbourhood. Close to the base the air may be cooler than at points more distant; the wave may therefore be propagated with less velocity in the vicinity of the base than at more distant points. In the second place it seems certain that the air behind the projectile will have a velocity gradient from the axis outwards. Near the axis the air will be moving faster than at more distant points.

Of these two effects the first will tend to diminish a, while the second will cause an increase in v in equation (2); the values of Ω will therefore be less near the apex of the wave.

The change of sign of Ω immediately behind the base is probably due to change in direction of the air's motion in the immediate neighbourhood. Lord Rayleigh has proved,* further, that such waves of condensation cannot be maintained in the absence of dissipative forces. It is therefore evident that some term involving the viscosity, such as vd/ν, must be included in the drag coefficient.

The Scale Effect

There is some experimental evidence of the dependence of the drag coefficient on d, and hence on some such term as vd/ν. For example, Cranz† quotes the following figures for the resistance per square centimetre of cross section deduced from Krupp's 1912 experiments:

(*a*) *For cylindrical shell* (flat heads).

Calibre (cm.)	for $v =$ 400	500	600	700	800 m./sec.
6·5	1·40	2·58	3·80	5·15	6·60 Kgm./cm².
10·0	1·29	2·20	3·30	4·70	6·30 „

* Loc. cit. † *Lehrbuch der Ballistik*, Vol. I (Berlin, 1925).

(*b*) *For ogival * shell*, 3 *calibres radius*.

Calibre (cm.)	for $v =$ 550	650	750	850 m./sec.
6	1·00	1·30	1·58	1·94 Kgm./cm².
10	0·98	1·25	1·52	1·85 ,,
28	0.62	0·81	1·01	1·25 ,,
30	—	—	0·90	1·06 ,,

In the absence of a term involving d, such as vd/ν, from the The coefficient the numbers in each vertical column would be equal. drag discrepancies are, no doubt, partly due to differences in the yaw of the projectiles, but they cannot be wholly accounted for in this way.

These results indicate that the drag per unit cross section (i.e. $4R/\pi d^2$) for projectiles of small calibre is larger than that for those of large calibre. Didion noticed this so-called scale effect as early as 1856† and deduced a relation between R/d^2 and d, but later he abandoned it as it would not hold for all velocities encountered in gunnery. This effect has not been confirmed in recent experiments and further evidence is needed before definite conclusions can be drawn.

Dependence of the Drag on Density

The assumption that R varies as ρ, other factors being constant, has considerable theoretical support, but up to the present the range of variation of ρ in experiments has been extremely small; we cannot therefore claim practical verification for this assumption. When more work has been done with the high-velocity air stream more light may be thrown on this question, as considerable variations of air density are easily obtained in this method.

The Function $f(v/a, vd/\nu)$

At the present time no satisfactory mathematical expression for the drag coefficient has been derived from theoretical considerations. We are therefore forced to accept values of the function derived from experiment alone. In ballistic calculations it has generally been assumed that the term vd/ν could be neglected, that is to say, that the drag coefficient is independent of the calibre; the function $f(v/a, vd/\nu)$ has, in consequence, been determined as a function of v/a only.

* See p. 360.

† *Lois de la résistance de l'air* (Paris, 1857).

Shape of Projectile

In our discussion we have so far considered the air resistance to projectiles of the *same* shape. Our next step is to consider the changes that occur in the resistance when the shape is altered.

All modern projectiles have a cylindrical body and a more-or-less pointed head. The head is usually ogival, that is to say, it is generated by the rotation of an arc of a circle about the axis of the projectile. The shape of the head is identified by the length of the radius (in calibres) of this arc. Thus in fig. 4 a head of 3-calibres radius is depicted.

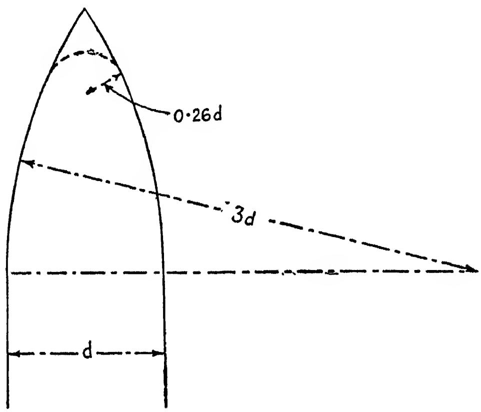


Fig. 4.—Shape of Head

When the point is rounded the radius of the rounding is also stated in calibres. The dotted head in fig. 4 would be described as a 3-calibres radius head with a 0·26-calibre rounded point.

In fig. 5 the drag co-efficients of a 15-c.m. projectile with flat-head and pointed heads of various lengths are plotted against the ratio, velocity of shell to velocity of sound. The curves are deduced from British and foreign experimental data.

The values of the drag co-efficient given in these curves may be used with any self-consistent system of units; for example, if the fundamental units used are the metre, kilogramme and second, these values of the drag coefficient when used in equation (1) will give the drag in metre-kilogramme-second units of force (1 unit = 100,000 dynes). Again, if the ft.-lb. and second be used, these values of the drag coefficient used in equation (1) will give the drag in poundals. This property arises, of course, from the fact that $f(v/a)$ has no physical dimensions.

The shape of the head, provided it is more-or-less pointed, does not appear greatly to influence the resistance at lower velocities. At velocities greater than about 350 m. per second, however, the effect of the length of the head is appreciable. At velocities greater than about 750 m. per second it appears that the shape of the point

is more important than that of the rest of the head. Thus the resistance is less, at these velocities, for a sharp-pointed 3-calibres radius head than for a 5·5 calibres radius head with a blunted point. For the same shape of point the resistance is less, at velocities greater than about 350 m. per second, for a long head (e.g. 5·5 calibres radius) than for a short one (e.g. 2-calibres radius).

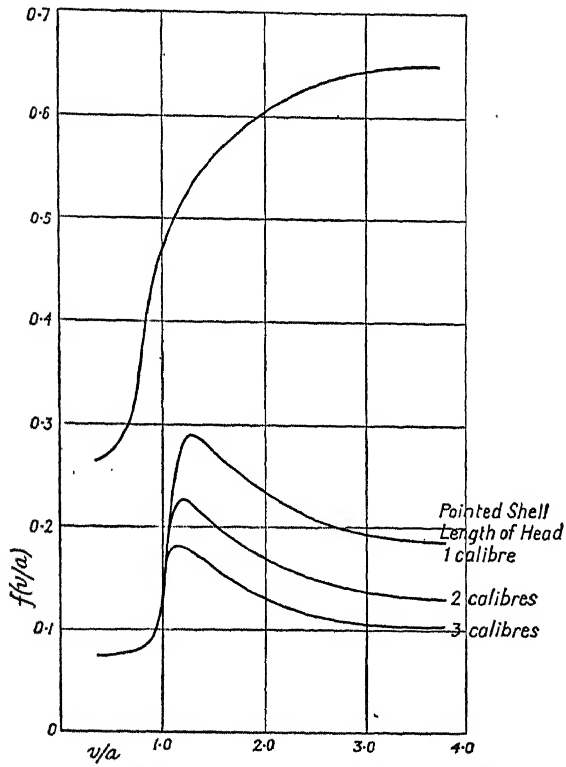


Fig. 5.—The Drag Coefficients for 15-cm. Projectile with Various Shapes of Head, plotted as a Function of Velocity

Experiments and trials lead to the conclusion that at high velocities a long head with a sharp point encounters considerably less resistance than a short head. For example an 8-calibres radius head experiences only about half the resistance of one of 2-calibres radius. Little advantage appears to be gained, however, by lengthening the head beyond 8-calibres ogival radius; thus 10- and 12-calibres radius heads are only slightly more effective than those of 8-calibres radius in reducing the air resistance.

The Base

The importance of the shape of the rear part of a body moving in a resisting medium has been realized for many years; the torpedo, the racing automobile, and the fusilage of an aeroplane are examples of "stream-lining" familiar to all. The suggestion has frequently been made that artillery projectiles should have a tapered (so-called "stream-line") base with a view to reducing the air resistance.

Experiments with rifle bullets have shown, however, that the stability is so seriously affected that any possible advantage gained by a pointed base is entirely eclipsed by the effects of a rapidly developing yaw.

In recent times a compromise has been effected in a shape known as the "boat-tail", which is illustrated in fig. 6. The base is tapered for a short distance and is then cut off square. The stability of the projectile is not appreciably affected by this modification of the base. Bullets of this shape were tried in France as far back as 1898, but they were found to have no great advantage over the flat base. It has, however, been shown recently that, although such a base has no particular advantage at high velocities, it has appreciable superiority over the flat base at velocities below about 450 m. per second. In the trials mentioned above, the French experimented at high velocity over short ranges and so failed to discover the merits of this shape.

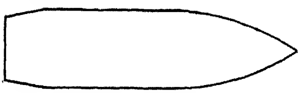


Fig. 6.—Boat-tail Projectile

Some extremely interesting and suggestive results are recorded by G. P. Wilhelm * of comparative experiments with bullets having the boat-tail and the flat base. The following table gives a summary of these results:

MUZZLE VELOCITY, 1500 FEET PER SECOND

Angle of Departure.	Range (Yards), Flat Base.	Range (Yards), Boat-tail.
0° 20′	200	220
0° 40′	360	410
1° 0′	500	580
1° 20′	630	720
1° 40′	740	840

* In "Long Range Small Arms Firing", Part VII, *Army Ordnance*, Washington, March–April, 1922.

MUZZLE VELOCITY, 2600 FEET PER SECOND

Angle of Departure.	Range (Yards), Flat Base.	Range (Yards), Boat-tail.
0° 20′	570	600
0° 40′	930	990
1° 0′	1050	1200
5° 0′	2250	2700
10° 0′	2900	3600
15° 0′	3250	4200
20° 0′	3500	4650

It is clearly seen that for velocities lower than 1500 ft. per second the boat-tail bullet considerably out-ranges the flat-base bullet. On the high-angle trajectories with a muzzle velocity of 2600 ft. per second the velocity of the bullet is, during the greater part of its flight, less than 1500 ft. per second, in fact it is generally only a few hundred feet per second; on the low-angle trajectories, on the other hand, the velocity is less than 1500 ft. per second only in the final stages of the flight. The appreciable gain in range by the boat-tail bullet fired with high muzzle velocity at high angles, and the small or negligible gain at low angles, thus confirms the hypothesis that the shape of the base is of greater importance at low than at high velocities.

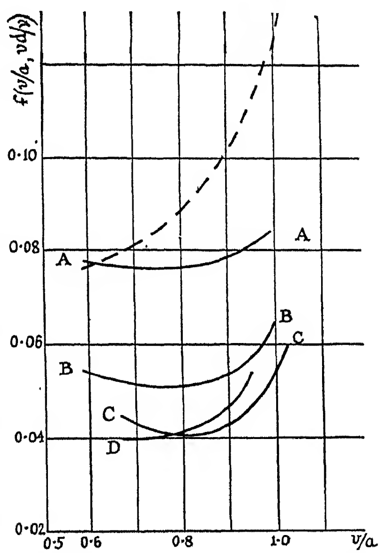


Fig. 7.—The Drag Coefficient deduced from High-speed Air-stream Experiments

Curve A.—For 5-cal.-rad. head and flat base. *Curve B.*—Same head as A, boat-tail base, taper, 5°. *Curve C.*—Same head as A, boat-tail base, taper, 7°. *Curve D.*—Same head as A, boat-tail base, taper, 9°. *Dotted Curve.*—The drag coefficient given in fig. 5 (flat base).

The results of experiments with the high-speed air stream are interesting in this connection. In fig. 7 some of the results of experiments conducted by the Ordnance Department of America are reproduced.*

* From "Experimental Determination of Forces on a Projectile", by G. F Hull, *Army Ordnance*, Washington, May–June, 1921.

These curves tend to show that for velocities below 350 m. per second the drag is greatly influenced by the shape of the base; on the other hand, as we have already seen, provided it is more or less pointed, the actual shape of the head has very little influence on the drag at these lower velocities.

From these results we may fairly conclude that the greater part of the air resistance to pointed projectiles at these velocities is due to the drag (suction) at the base, and that this drag is appreciably reduced by boat-tailing.

The divergence of curve A in fig. 7 from the dotted curve is not clearly understood. It is possible that the assumptions made in deducing the velocity of the air stream are not altogether sound and lead to values which are too high; it is also possible that the rod supporting the model (p. 354) may materially affect the air flow at the base and so modify the drag.

We have seen that at high velocities the drag is greatly affected by the shape of the head, whereas no appreciable effect is produced by modifying the base. The probable explanation of this is not far to seek. At velocities greater than 750 m. per second (the so-called "cavitation" velocity of air) the vacuum at the base must be of high order, and, as the velocity of the projectile increases, it must tend asymptotically to a perfect vacuum. We should therefore expect that the component of the air resistance due to the base is tolerably constant at these high velocities, whereas the total resistance is rapidly increasing with velocity. The component due to the base, with increasing velocity, soon becomes a small part of the total resistance, and therefore any possible modification of it, due to shape of base, can have little influence on the whole.

Our observations on the effect of shape of pointed projectiles may now be conveniently summarized. At velocities less than about 350 m. per second the drag at the base contributes the greater part of the air resistance, so that the shape of the base is of greater importance. At velocities between about 350 m. and 750 m. per second we have an intermediate stage in which the shape of the head gradually gains ascendancy. At velocities greater than about 750 m. per second the greater part of the resistance is due to the head, and the shape of the latter is of greater importance than the shape of the base.

Before leaving the subject of shape we must refer to some extremely interesting experiments designed to determine the pressure distribution on the head of a projectile moving with high velocity.

The Pressure Distribution on the Head of a Projectile

The pressure at any point of a body moving through a fluid consists of two components—the *static pressure*, which is the pressure of the fluid when the body is at rest, and the *dynamic pressure*, which is due to the motion. The sum of these two at any point is the total pressure at that point and is essentially positive; the dynamic pressure may be either positive or negative.

A series of experiments was carried out by Bairstow, Fowler and Hartree to determine the dynamic pressure at various points on the head of a shell moving with high velocity.* The fundamental idea of the experiments is the use of a service time-fuze † as a manometer to determine the pressure under which the powder is burning.

Projectiles were fitted with hollow caps which entirely enclosed the fuze; each cap had a number of holes drilled in it at the same distance from the nose; the pressure on the fuze was thus practically equal to that at the holes.

The projectiles were fired along the same trajectory at various fuze settings and times to burst were observed; a relation between fuze setting and time was thus obtained, whence was deduced a relation between rate of burning and time.

Since rate of burning is a function of the pressure on the fuze, by comparison with laboratory experiments it is possible to convert this relation to one of pressure and time. Knowing the velocity of the projectile at various times of flight on the trajectory, it is thus possible to deduce the pressure in terms of the velocity.

By repeating the experiment with other caps of the same size and shape with holes at other distances from the nose, the pressure distribution over the head is obtained at a number of velocities.

The results of the experiments are reproduced in fig. 8. The ordinates are values of $p/\rho v^2$; as this quantity has no physical dimensions the values given may be used with any self-consistent system of units. The abscissæ are distances from the nose of the projectile. Observations were made at four positions on the head, indicated by

* For a full account of these experiments see "The pressure distribution on the head of a shell moving at high velocities", *Proc. Roy. Soc.* A, 97, 1920.

† "A service time-fuze contains a train of gunpowder, which is ignited by a detonator pellet on the shock of discharge from the gun. The 'setting' of the fuze can be varied so that a length of powder train depending on the setting is burnt before the magazine of the fuze is ignited and the shell exploded. The setting is specified by a number which defines the length of composition burnt on an arbitrary scale. The time of burning is taken as equal to the time interval between the firing of the gun and the bursting of the shell."—Loc. cit.

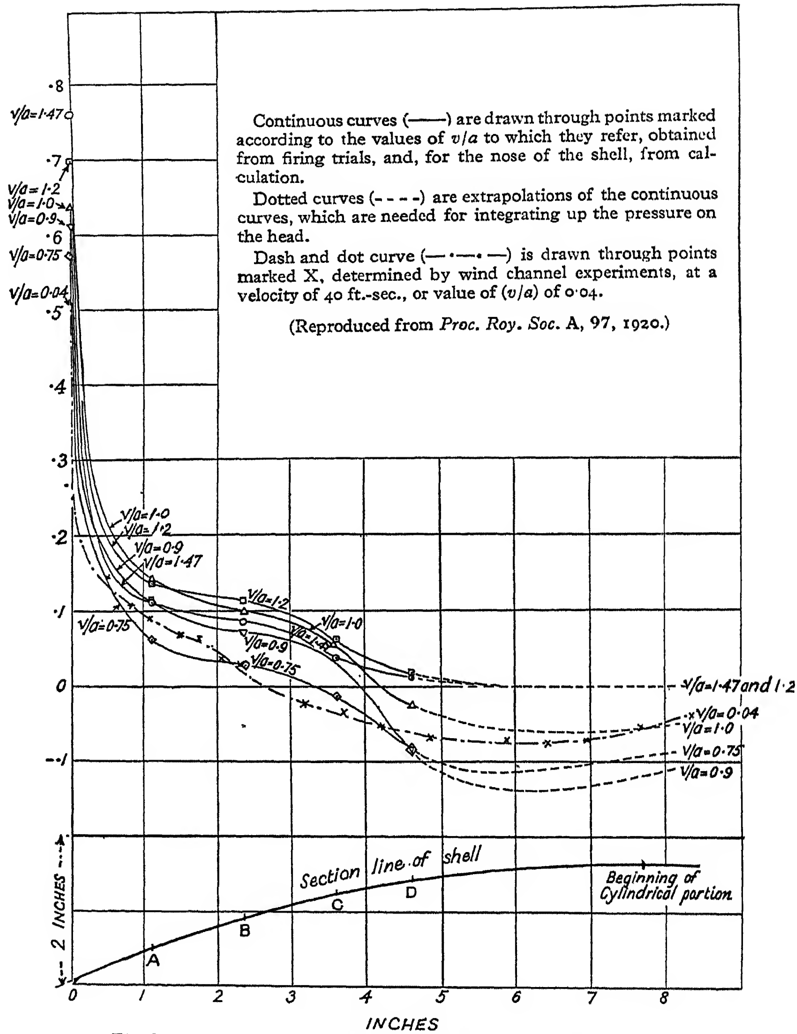


Fig. 8.—Distribution of Pressure on the 6-cal.-rad. Head of a 3·3-in. Shell

the points A, B, C, and D. The value of $p/\rho v^2$ at the nose was calculated from Rayleigh's formula * in each case.

$$*\ \frac{p}{\rho v^2} = \frac{a^2}{\gamma v^2}\left[\left\{1 + \tfrac{1}{2}\left(\gamma - 1\right)\frac{v^2}{a^2}\right\}^{\frac{\gamma}{\gamma - 1}} - 1\right] \text{ when } \frac{v}{a} \leqq 1;$$

$$\frac{p}{\rho v^2} = \frac{1}{\gamma}\left[\frac{\{\tfrac{1}{2}(\gamma + 1)\}^{\gamma + 1}}{\gamma - \tfrac{1}{2}(\gamma - 1)a^2/v^2}\right]^{\frac{1}{\gamma - 1}} - \frac{a^2}{\gamma v^2} \text{ when } \frac{v}{a} \geqq 1;$$

deduced from the formulæ given in his *Scientific Papers*, Vol. V, p. 610.

These curves reveal most emphatically the necessity of a sharp point at the nose of the projectile. Compared with the pressure encountered at the nose the pressures at other points of the head are quite small.

The authors integrated numerically the observed pressures over the head in each case, and derived values of the drag coefficient for

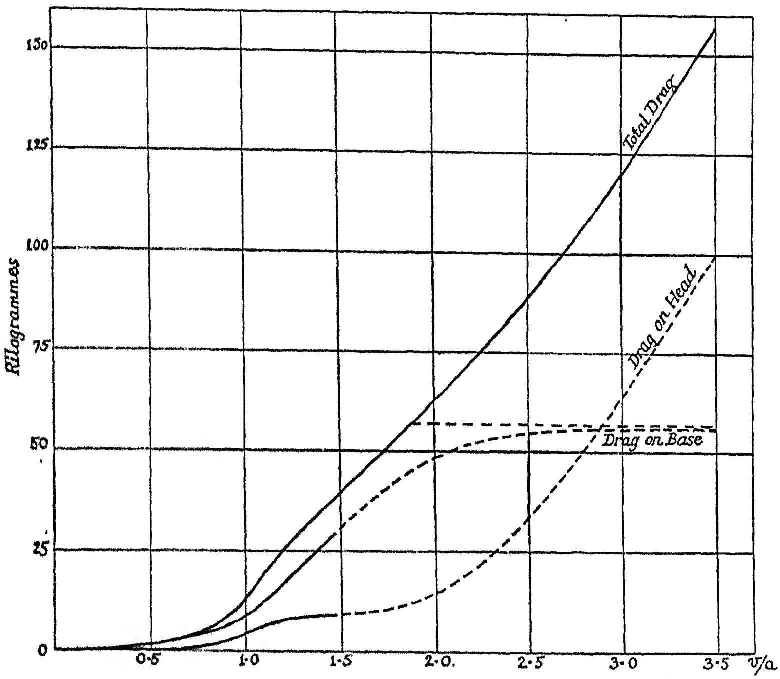


Fig. 9.—Drag in Kilogrammes Weight on a 3·3-in. 6 C.R.H. Projectile

the dynamic resistance on the head. From these results we have computed the actual dynamic resistance on the head; it is plotted against velocity in fig. 9.

The total drag on projectiles of this shape is also shown (approximately) in the figure. By subtracting the head resistance from the total resistance we have derived an approximate curve of the drag at the base. The horizontal dotted line indicates the drag due to a complete vacuum at the base in this case, and the dotted extensions of the curves represent a tentative extrapolation of the results of the experiments.

The Effect of Yaw on the Drag

Before approaching the complicated reaction of the air to a yawing projectile it will be convenient to consider, briefly, the effect of yaw on the drag; we now define the latter as the force exerted on the projectile in the opposite direction to the relative motion of the air and the centre of gravity of the projectile.

If δ be the angle of yaw the drag coefficient will now take the form

$$f(v/a,\ vd/\nu,\ \delta),$$

and, in this notation, the drag previously considered takes the form *

$$f(v/a,\ vd/\nu,\ 0).$$

The manner in which the drag coefficient varies with yaw at very low velocities has been determined experimentally in wind channels. The results of such experiments on a 3-in. projectile with a 2-calibres-radius head and 0·15-calibre rounded point are given in fig. 10;† the ordinates are values of the ratio

$$f(v/a,\ vd/\nu,\ \delta)\Big/f(v/a,\ vd/\nu,\ 0).$$

The velocity at which the experiments were conducted was 40 ft. per second ($v/a = 0{\cdot}04$).

We have seen that the drag on a body moving in air is approximately proportional to the square of the velocity,‡ provided that the shape is such that the condensation in front is of small amplitude; for example, this quadratic law holds for pointed projectiles for values of v/a not greater than 0·65 when the yaw is zero.

We might reasonably expect the quadratic law to hold for yawing pointed projectiles within the same limit of velocity, provided that the yaw is such that the air is encountered point first; for the condensation would be of the same order as when the yaw is zero. Within this limit for yaw, with values of v/a less than 0·65, we should therefore expect that the ratio $f(v/a, vd/\nu, \delta)/f(v/a, vd/\nu, 0)$ is independent of v

* Approximately. Except in the case of results quoted from air-stream experiments we cannot be certain that the values of the drag coefficient hitherto used are for zero yaw. All we can affirm is that they are the values for very small or zero yaw.

† This curve is derived from one given in "The Aerodynamics of a Spinning Shell", by Fowler, Gallop, Lock, and Richmond, *Phil. Trans.* A, **591**, 1920.

‡ Except, of course, for such low velocities that the drag is due to viscosity alone.

and therefore a function of δ only. The limit of δ for the projectile under consideration appears to be about 45°.

Experiments with the high-velocity air stream tend to verify the independence of velocity of this ratio within the limits mentioned,

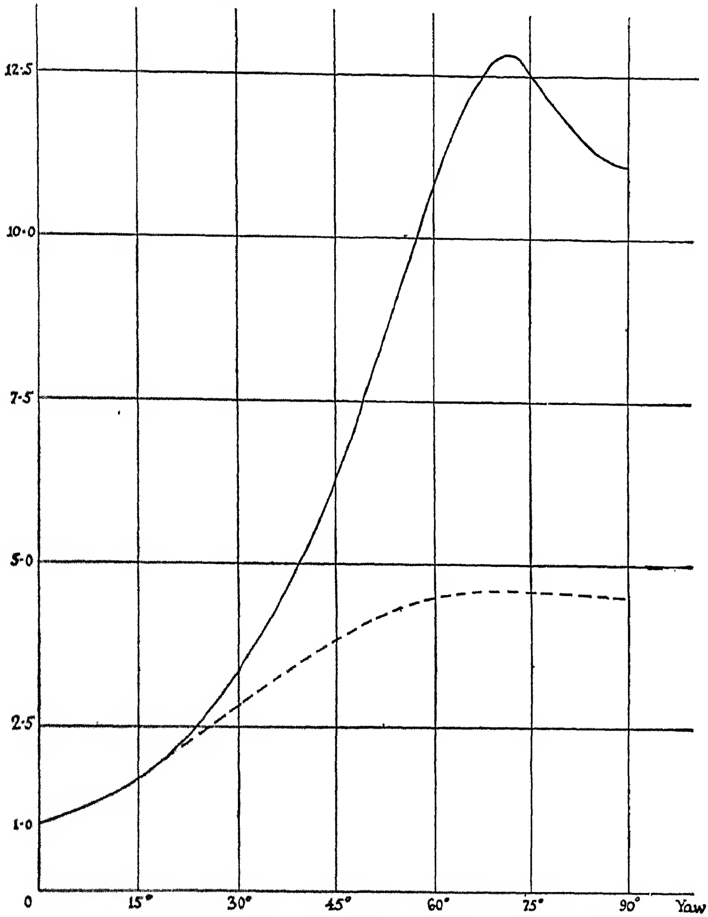


Fig. 10.—The Ratio $f(v/a, vd/\nu, \delta)/f(v/a, vd/\nu, 0)$ plotted against the yaw δ for $v/a = 0{\cdot}04$

but at present the number of results available is insufficient to justify our drawing definite conclusions.

The dotted curve in fig. 10 gives, approximately, the ratio between the total plane areas encountered by the projectile with yaws δ and zero. The former is, of course, equal to the area of the shadow cast

by the projectile, in a parallel beam of light inclined at angle δ with the axis, upon a plane normal to the beam; the latter is simply the cross-sectional area of the projectile. The two curves in the figure are coincident for small angles of yaw, but rapidly diverge as the yaw increases; both curves appear to have a maximum at about the same value of the yaw.

We have at present no knowledge of the effect of yaw on the drag at higher velocities; when more work has been done with the high-velocity air stream it is hoped that our knowledge in this direction will have been considerably extended.

The Drag Coefficient; Concluding Remarks

In our consideration of the drag coefficient we have limited the number of arguments of the function to three only, namely v/a, vd/ν, and δ. There appear to be two other possible arguments, namely γ, the ratio of the specific heats, and σ/d, where σ is the effective diameter of the molecules of the air.

Variations of γ are so small in practice that no evidence of its effect is available. Expressions deduced from thermodynamic theory, by various authors, for the drag in one-dimensional motion may give an indication of the manner in which it occurs.*

There is at present no evidence of the necessity of the argument σ/d. If further experimental results show that the argument vd/ν does not adequately account for the " scale " effect (e.g. with varying ν), it will of course be necessary to include some other argument involving d, such as σ/d.

Finally, in the case of a projectile moving in air, as distinct from one which is stationary in an air stream of constant velocity, there is the question of retardation. It is just possible that some such argument as rd/v^2, r being the retardation of the projectile, may be required to co-ordinate the results of air-stream experiments with those of experiments on a projectile moving in air; but it is difficult to see how the retardation can ever have an appreciable effect on the drag, except, perhaps, when the velocity is in the neighbourhood of the velocity of sound in air.

* See for example the footnote on p. 346; also Vieille, *Comptes Rendus*, 130 (1900), and Okinghaus, *Monatsh. für Mathem. u. Phys.*, **15** (1904).

REACTION TO A YAWING, SPINNING PROJECTILE

The most complete specification in existence of the system of forces acting on a projectile is that given by Fowler, Gallop, Lock, and Richmond in "The Aerodynamics of a Spinning Shell", *Phil. Trans.* A, **591** (1920). In this paper the authors describe experiments conducted to determine numerically the principal reactions, other than the drag, to which a spinning shell is subjected.

The experiments are confined to the study of the angular oscillations of the axis of the shell relative to the direction of motion of the centre of gravity. The projectile is fired horizontally through a series of cardboard targets fixed, vertically, along the range at 30 ft. and 60 ft. intervals. Initial disturbances at the muzzle give rise to oscillations of the projectile of sufficient amplitude for measurement; the details of the oscillations are obtained by measuring the shape of the holes made in the cards. If, on passing through a card, the shell is yawing, the resulting hole will be elongated; the length of the longer axis of the hole determines the yaw; the orientation of this axis determines the azimuth of the plane containing the axis of the shell and the direction of motion; these two angles determine the direction of the shell's axis completely.

The range containing the cards is so short, and the velocity so high, that the effect of gravity is negligible. If, then, we ignore damping forces, the angular motions of the axis of a top and the axis of a shell are identical, provided that (1) the top and shell have the same axial spin and axial moment of inertia; (2) the transverse moment of inertia of the top about its point of support is equal to the transverse moment of inertia of the shell about its centre of gravity; (3) the moment of gravity about the point of the top is equal to the moment of the force system on the shell about its centre of gravity. The formal solutions of the two problems are then identical.

From the periods of the oscillations of the axis of the top we can deduce the moment of the disturbing couple, and *vice versa*; similarly the moment of the force system on the shell can be deduced. The damping forces can then be determined from the nature of the decay of the oscillations.

The reactions are described by the authors as follows:

The Principal Reactions

When the shell, regarded for the moment as without axial spin, has a yaw δ, and the axis of the shell OA and the direction of motion OP remain in the same relative positions, the force system can by symmetry be represented, as shown in fig. 11, by the following components, specified according to aerodynamical usage:

(1) *The drag* R acting through the centre of gravity O, in the direction PO.

(2) A component L, at right angles to R, called the *cross-wind force*, which acts through O in the plane of yaw POA, and is positive when it tends to move O in the direction from P to A.

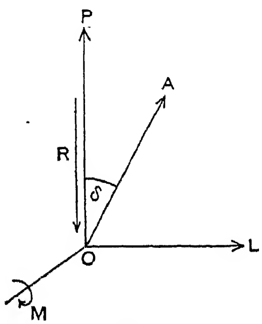


Fig. 11.

(3) A moment M about O, which acts in the plane of yaw, and is positive when it tends to increase the yaw.

The following forms are assumed for L and M:

$$L = \rho v^2 d^2 \sin\delta\, f_L(v/a, \delta).$$

$$M = \rho v^2 d^3 \sin\delta\, f_M(v/a, \delta).$$

These equations are of the most natural forms to make f_L and f_M of no physical dimensions. The form chosen is suggested by the aerodynamical treatment of the force system on an aeroplane. Since L and M, by symmetry, vanish with δ, the factor $\sin\delta$ is explicitly included in these expressions in order that f_L and f_M may have non-zero limits as $\delta \to 0$.

The Damping Reactions

The yawing moment due to yawing.—In practice the direction of the axis of the shell, relative to the direction of motion, changes fairly rapidly. By analogy with the treatment of the motion of an aeroplane, we assume, tentatively, that the components of the force system R, L, and M are unaltered by the angular velocity of the axis, but that the effect of the angular motion of the latter can be represented by the insertion of an additional component, namely, a couple H, called *the yawing moment due to yawing*, which satisfies the equation

$$H = \rho v w d^4 f_H(v/a, \ldots),$$

where w is the resultant angular velocity of the axis of the shell. The form is chosen to make f_H of no physical dimensions and is the only one suitable for the purpose.

The couple is assumed to act in such a way as directly to diminish w (see fig. 12). It is suggested by, and is analogous to, the more important of the "rotary derivatives" in the theory of the motion of an aeroplane.

The coefficient f_H may be expected to vary considerably with v/a, and it may depend appreciably on other arguments such as wd/v and δ.

The effect of the axial spin.—The spin N gives rise to certain additional components of the complete force system.

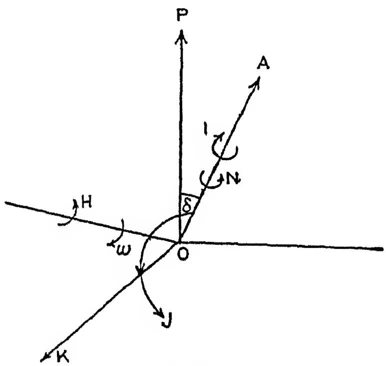


Fig. 12

There will be a couple I which tends to destroy N, and, when the shell is yawing, a sideways force, which need not act through the centre of gravity, analogous to that producing swerve on a golf or tennis ball. This force must, by symmetry, vanish with the yaw; it is assumed to act normal to the plane of yaw (any component it may have in the plane of yaw is inevitably included in either R or L). The complete effects of the spin N can therefore be represented by the addition to the force system of the couples I and J and the force K, acting as shown in fig. 12.

To procure the correct dimensions we may assume that these reactions have the forms

$$\mathrm{I} = \rho v \mathrm{N} d^4 f_{\mathrm{I}}.$$

$$\mathrm{J} = \rho v \mathrm{N} d^4 \sin\delta\, f_{\mathrm{J}}.$$

$$\mathrm{K} = \rho v \mathrm{N} d^3 \sin\delta\, f_{\mathrm{K}}.$$

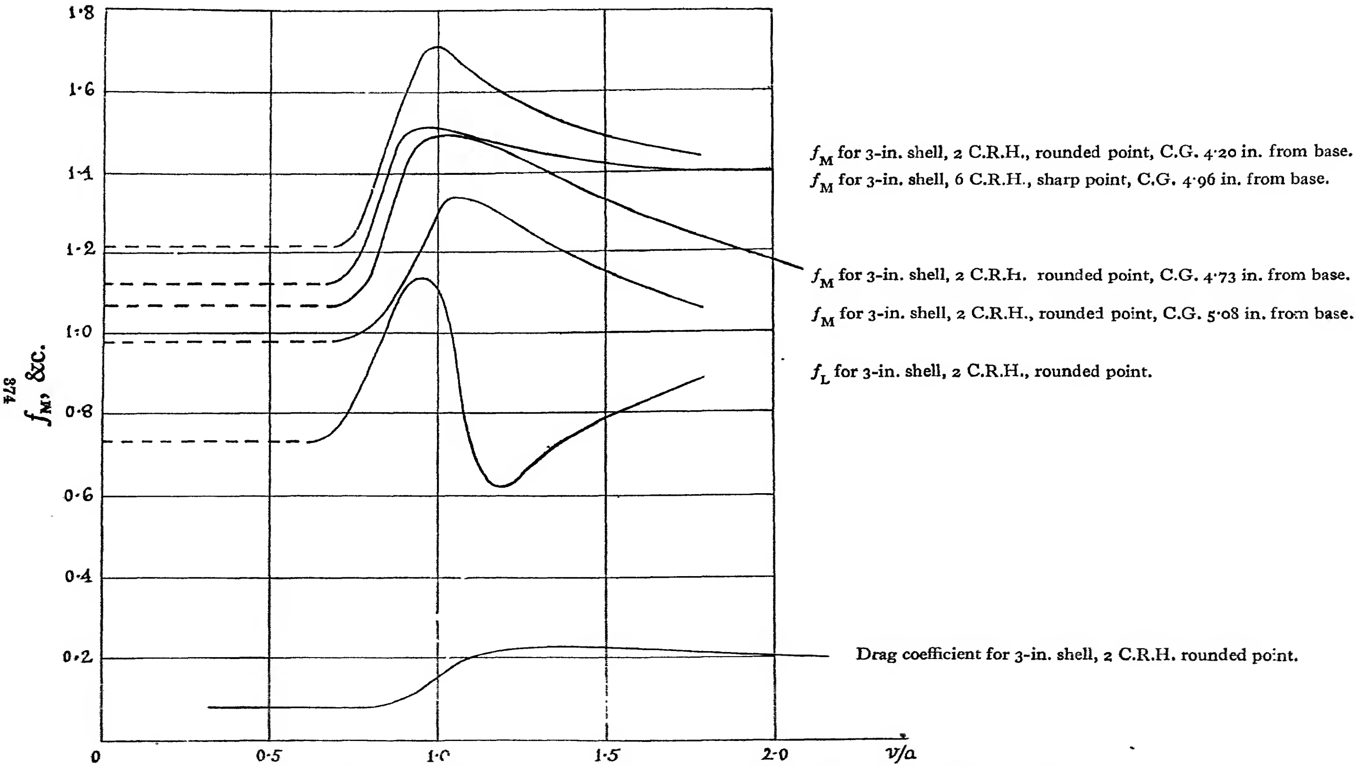


Fig. 13.—$f_M(v/a, 0)$, $f_L(v/a, 0)$ and the Drag Coefficient plotted against v/a. (From *Phil. Trans.* A, 591, 1920)

The coefficients f_I, f_J, f_K may depend effectively on a number of variables which we can make no attempt to specify in the present state of our knowledge.

It will be seen that this specification is equivalent to a complete system of three forces and three couples referred to three axes at right angles. Owing to the complex nature of the reactions the authors considered it essential to construct the specified system

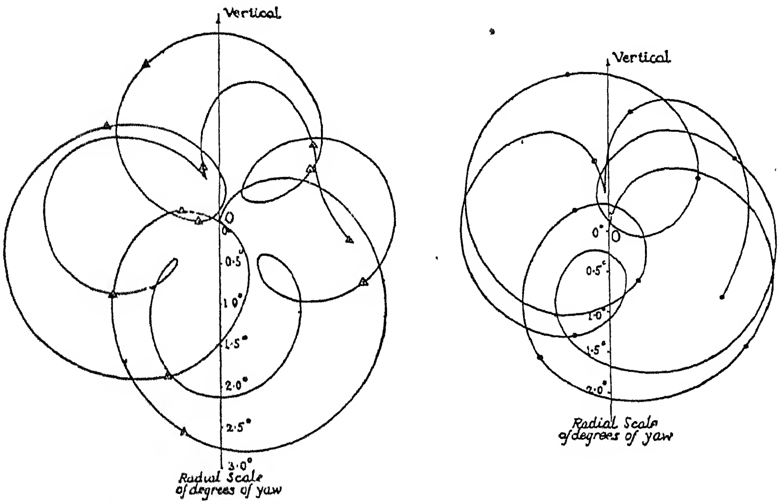


Fig. 14.—Examples of Path of Nose of Shell relative to the Centre of Gravity

(From *Phil. Trans.* A, 591, 1920)

instead of attempting to analyse a complete system of three forces and three couples and to assign each component to its proper causes.

The experiments were designed to determine L * and M and to give an indication of the magnitude of the chief damping couple H. It was not possible to determine I, J, and K; all three are presumably small compared with H and very small compared with L and M; no certain evidence that they exist was given by the experiments.

The results of the experiments are exhibited in fig. 13, which is reproduced from the paper. The units in which the coefficients are expressed are suitable for use with any self-consistent system. The

* Values of L were deduced from values of M for shell of the same external shape, with centres of gravity at different positions along the axis of figure.

curve of the drag coefficient is also given; in comparing this with f_L and f_M it must be remembered that the latter should be multiplied by $\sin\delta$. The shape of the curve for f_L is rather unexpected; the curves for f_M appear to exhibit the same tendencies as that for the drag coefficient.

The values of f_H deduced from the experiments were rough; they varied from about 1·4 to 5·0. It was impossible to deduce any details concerning the variations of f_H with velocity, but the right order of magnitude is represented by these limits. In comparing f_H with f_M it must be remembered that the former is multiplied by wd, whereas the latter is multiplied by the much larger quantity v.

The general features of the motion of the axis of the shell and of the damping are shown in the examples, reproduced from the paper cited, in fig. 14.

SUBJECT INDEX

(*See also Name Index*)

NAME INDEX